Lax structures in 2-category theory

Miloslav Štěpán

Faculty of Science, Masaryk University

Supervisor: John Bourke

March 18, 2025

Plan of the talk:

- Background
- Results
- Future work
- Publications and talks

Background

The thesis is a contribution to category theory:

- 2-category theory,
- 2-dimensional monad theory,
- codescent objects.

The goal was to prove **lax analogues** of results on **pseudo** structures.

2-categories

2-categories & **double categories**:

$$
\begin{tikzcd}
a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, squiggly, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\alpha"] & b \arrow[d, "v"] \\
c \arrow[r, squiggly, "h"'] & d
\end{tikzcd}
$$

Sets with additional structure (posets, groups, vector spaces) form categories.

Categories with additional structure (monoidal categories, cocomplete categories) form 2-categories.

2-monads

A monad (T, μ, η) on Set consists of:

- a functor T : Set $\to$ Set,
- a natural transformations $\mu : T^2 \Rightarrow T$,
- a natural transformations $\eta : 1 \Rightarrow T$,

satisfying the following identities:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

2-monads

A 2-monad (T, μ, η) on Cat consists of:

- a 2-functor $T : \mathrm{Cat} \to \mathrm{Cat}$,
- a 2-natural transformations $\mu : T^2 \Rightarrow T$,
- a 2-natural transformations $\eta : 1 \Rightarrow T$,

satisfying the following identities:

$$
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}
$$

Important 2-monads

2-monads of form $\mathsf{Cat}(T)$:

- well-behaved class of 2-monads built out of ordinary monads
- Example: T free monoid monad $\rightsquigarrow$ $\mathsf{Cat}(T)$ free (strict) monoidal category 2-monad

lax-idempotent 2-monads:

- introduced in [Koc95], [Zöb76],
- they satisfy $\mu \dashv \eta T$,
- Example: free cocompletion 2-monads.

Levels of strictness

Algebras for 2-monads admit three levels of strictness.

Level	Associativity	Example
strict	$(a * b) * c = a * (b * c)$	a group G
pseudo	$(A \times B) \times C \cong A \times (B \times C)$ $(U \otimes_{\mathbb{R}} V) \otimes_{\mathbb{R}} W \cong U \otimes_{\mathbb{R}} (V \otimes_{\mathbb{R}} W)$	sets vector spaces
lax colax	$(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ $(A \otimes B) \otimes C \leftarrow A \otimes (B \otimes C)$	set difference

For a 2-monad T, the categories of algebras will be denoted by:

T-Alg Ps-T-Alg Lax-T-Alg Colax-T-Alg

Morphisms of algebras can also be **strict**, **pseudo**, **(co)lax** – we use the lower index for this, e.g.:

$$\text{Colax-T-Alg}_{\text{ps}} .$$

In the literature, there is often a preference for pseudo over lax.

Results

Question 1 - making colax sturctures strict

Theorem, [Lac02]

Let T be a 2-monad on a 2-category.

- The inclusion below has a left adjoint if and only if $\text{T-Alg}_{\text{strict}}$ admits **codescent objects**.

$$\begin{tikzcd} \text{T-Alg}_{\text{strict}} \arrow[rr, hook, bend right] \arrow[rr, phantom, "\perp"] & & \text{Ps-T-Alg}_{\text{ps}} \arrow[ll, dashed, bend right, "(-)'"'] \end{tikzcd}$$

- If T preserves them, each $\mathbb{A} \in \text{Ps-T-Alg}_{\text{ps}}$ is canonically equivalent to $\mathbb{A}'$.

Implies coherence for monoidal categories, bicategories, pseudofunctors. In [Lac14] and [Ště20], a version involving $\text{Colax-T-Alg}_{\text{lax}}$ instead of $\text{Ps-T-Alg}_{\text{ps}}$ has been proven.

Question 1 - making colax sturctures strict

Theorem, [Lac02]

Let T be a 2-monad on a 2-category.

- The inclusion below has a left adjoint if and only if $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{strict}}$ admits **codescent objects**.

$$\begin{tikzcd} \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{strict}} \arrow[rr, hook, bend right] \arrow[rr, phantom, "\perp"] & & \mathsf{Ps}\text{-}\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{ps}} \arrow[ll, dashed, bend right, "(-)'"'] \end{tikzcd}$$

- If T preserves them, each $\mathbb{A} \in \mathsf{Ps}\text{-}\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{ps}}$ is canonically equivalent to $\mathbb{A}'$.

Question 1: What are the examples of the colax version?

Answer

Theorem 4.4.3

Let T' be a cartesian monad on a category $\mathcal{E}$ with pullbacks. Consider $T := \mathsf{Cat}(T')$ on $\mathsf{Cat}(\mathcal{E})$.

- The inclusion of strict algebras to colax ones admits a left adjoint:

$$\begin{tikzcd} \mathsf{T\text{-}Alg}_{\mathsf{strict}} \arrow[rr, hook, bend right] \arrow[rr, phantom, "\perp"] & & \mathsf{Colax\text{-}T\text{-}Alg}_{\mathsf{lax}} \arrow[ll, dashed, bend right, "(-)^{\dagger}"'] \end{tikzcd}$$

- Each $\mathbb{A} \in \mathsf{Colax\text{-}T\text{-}Alg}_{\mathsf{lax}}$ is related to $\mathbb{A}^{\dagger}$ via a canonical adjunction.

Answer – remarks

- Applies for instance to (unbiased) colax monoidal categories or colax functors.
- Done by studying and generalizing constructions from [Web15].
- A double category-like structure has been introduced – a *codomain-colax category*.
- Note – no colimit assumptions on $\mathcal{E}$ or $\mathrm{Cat}(\mathcal{E})$.

By-product: a result on factorization systems

Theorem 3.3.33

- Any category $\mathcal{C}$ with two classes of morphisms $(\mathcal{E}, \mathcal{M})$ can be assigned a double category $D_{\mathcal{E},\mathcal{M}}$.
- Any double category X can be assigned a category $\mathsf{Cnr}(X)$ with two classes of morphisms $(\mathcal{E}_X, \mathcal{M}_X)$.

These establish an equivalence between **orthogonal factorization systems** and certain **double categories**:

$$\begin{tikzcd} \mathcal{OFS} \arrow[r, shift right=1.5ex, "D"'] \arrow[r, phantom, "\simeq"] & \mathsf{FactDbl} \arrow[l, shift right=1.5ex, "\mathsf{Cnr}(-)"'] \end{tikzcd}$$

Analogous results hold for **strict** and **weak** factorization systems (Theorem 3.3.7, Corollary 4.2.17).

By-product: a result on factorization systems

Theorem 3.3.33

- Any category $\mathcal{C}$ with two classes of morphisms $(\mathcal{E}, \mathcal{M})$ can be assigned a double category $D_{\mathcal{E},\mathcal{M}}$.
- Any double category X can be assigned a category $\mathsf{Cnr}(X)$ with two classes of morphisms $(\mathcal{E}_X, \mathcal{M}_X)$.

These establish an equivalence between **orthogonal factorization systems** and certain **double categories**:

$$\mathcal{OFS} \underset{D}{\overset{\mathsf{Cnr}(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathsf{FactDbl}$$

The philosophy: A **structure** on a 1-dimensional object is replaced by a **property** of a 2-dimensional object.

Question 2 – colimits for lax morphisms

Limits in $\text{T-Alg}_{\text{lax}}$ understood – see [Lac05], [LS12].
Colimits, not so much. For **pseudo morphisms**, we have:

Theorem [BKP89, 5.1, 5.8]

Let T be a 2-monad on a 2-category. Assuming $(\dots)$, we have:

- Any 2-adjunction as below induces a biadjunction:

$$\begin{tikzcd} \text{T-Alg}_{\text{strict}} \arrow[r, "J"'] & \text{T-Alg}_{\text{ps}} \arrow[r, "G"'] \arrow[r, phantom, bend right=60, "\perp"'] & \mathcal{L} \arrow[ll, bend right=40, "H"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \text{T-Alg}_{\text{ps}} \arrow[r, "G"'] \arrow[r, phantom, bend left=50, "\perp"] & \mathcal{L} \arrow[l, bend right=60, "JH"'] \end{tikzcd}$$

- If $\text{T-Alg}_{\text{strict}}$ is 2-cocomplete, T-Alg_{ps} is bicocomplete.

Question 2: What is the version involving **lax** instead of **pseudo**?

Question 2 – observations

- Need to replace biadjunctions by something weaker – *colax adjunctions* of [Gra74].
- Need to replace bicolimits by something weaker – a special case of *enriched weak colimits* of [LR12] ($\mathcal{V} = \mathsf{Cat}, \mathcal{E} := \{$reflectors$\}$). Introduced in [Gra74], one example appears in [Mil03].
- The theorem in [BKP89] is a corollary of a **very formal** result.

Question 2 – intermediate result

‘Big Colax Adjunction Theorem’ 5.3.15, 25

Let D be a lax-idempotent 2-monad on a 2-category $\mathcal{K}$.

- Any 2-adjunction as below induces a colax adjunction:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "J"'] & \mathcal{K}_D \arrow[r, "G"'] \arrow[d, phantom, ""] & \mathcal{L} \arrow[ll, bend right=50, "H"'] \arrow[ll, phantom, bend right=25, "\top"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{K}_D \arrow[r, "G"'] & \mathcal{L} \arrow[l, bend right=60, "JH"'] \arrow[l, phantom, bend right=30, "\top"] \end{tikzcd}$$

- If $\mathcal{K}$ is 2-complete, $\mathcal{K}_D$ is weakly complete.

Done by exploiting the *left Kan pseudomonad* presentation [MW12].
Also needed to generalize [Bun74] from 2-functors to pseudofunctors.

Question 2 – intermediate result

‘Big Colax Adjunction Theorem’ 5.3.15, 25

Let D be a lax-idempotent 2-monad on a 2-category $\mathcal{K}$.

- Any 2-adjunction as below induces a colax adjunction:

$$
\begin{tikzcd}
\mathcal{K} \arrow[r, "J"'] & \mathcal{K}_D \arrow[r, "G"'] & \mathcal{L} \arrow[ll, bend right, "H"'] \arrow[ll, phantom, bend right=20, "\top"]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathcal{K}_D \arrow[r, "G"'] & \mathcal{L} \arrow[l, bend right=60, "JH"'] \arrow[l, phantom, bend right=30, "\overline{\top}"]
\end{tikzcd}
$$

- If $\mathcal{K}$ is 2-complete, $\mathcal{K}_D$ is weakly complete.

Application: new results on the bicategory Prof.

Question 2 – answer

Theorem 5.4.9, 11

Let T be a 2-monad on a 2-category. Assuming $(\dots)$, we have:

- Any 2-adjunction as below induces a colax adjunction:

$$\begin{tikzcd} \text{T-Alg}_{\text{strict}} \arrow[r, "J"'] & \text{T-Alg}_{\text{lax}} \arrow[r, "G"'] \arrow[l, phantom, ""] & \mathcal{L} \arrow[ll, bend right, "H"', "\perp"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \text{T-Alg}_{\text{lax}} \arrow[r, "G"'] & \mathcal{L} \arrow[l, bend right, "JH"', "\perp"] \end{tikzcd}$$

- If $\text{T-Alg}_{\text{strict}}$ is 2-cocomplete, $\text{T-Alg}_{\text{lax}}$ is weakly cocomplete.

Examples

Monoidal categories & lax monoidal functors.
Categories with J-colimits & **all** functors between them.

Other results

"Lax-idempotentiation process"

Fakir [Fak70] described a process that turns a monad on a category into an idempotent one. He gives a right adjoint to the inclusion:

$$\mathrm{Idempotent}(\mathcal{C}) \hookrightarrow \mathrm{Monad}(\mathcal{C})$$

Given a monad T on $\mathcal{C}$, using equalizers we may construct a transfinite sequence of monads and monomorphisms:

$$T \hookleftarrow T_+ \hookleftarrow T_{++} \hookleftarrow \dots$$

Chapter 6 provides the first steps towards generalizing this to **lax-idempotent** 2-monads on 2-categories:

- Proposition 6.1.14: T is lax-idempotent if and only if $T = T_+$.
- Theorem 6.1.13: if T preserves descent objects, T_+ is lax-idempotent.

Application to lax morphisms

Let T be a 2-monad on $\mathcal{K}$. Assuming (...), we have:

$$
\begin{tikzcd}
\text{T-Alg}_{\text{strict}} \arrow[rr, bend right, "U^T"'] \arrow[rr, phantom, "\perp"] & & \mathcal{K} \arrow[ll, bend right, "F^T"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{T-Alg}_{\text{strict}} \arrow[rr, hook, bend right] \arrow[rr, phantom, "\perp"] & & \text{T-Alg}_{\text{lax}} \arrow[ll, bend right, "(-)^{\dagger}"']
\end{tikzcd}
$$

They generate comonads $F^T U^T$ and Q_l on $\text{T-Alg}_{\text{strict}}$.

Theorem 6.1.15

The comonad Q_l is the reflection of $F^T U^T$ along the inclusion:

$$\text{Lax-idempotent}(\text{T-Alg}_{\text{strict}}) \hookrightarrow \text{2-Comonad}(\text{T-Alg}_{\text{strict}})$$

Note: observed by Richard Garner [Gar18].

The relationship with multicategories

Let $\widetilde{T}$ be a cartesian monad on a category $\mathcal{E}$ with pullbacks. Consider $T := \mathrm{Cat}(\widetilde{T})$ on $\mathrm{Cat}(\mathcal{E})$. We have an adjunction that generates the comonad Q_l on $\text{T-Alg}_{\text{strict}}$:

$$\begin{tikzcd} \text{T-Alg}_{\text{strict}} \arrow[rr, hook, bend right] \arrow[rr, phantom, "\perp"] & & \text{T-Alg}_{\text{lax}} \arrow[ll, bend right, "(-)^{\dagger}"'] \end{tikzcd}$$

Theorem 6.2.5

Q_l-coalgebras are equivalently $\widetilde{T}$-multicategories.

A special case appeared in [DPP06, Theorem 2.13].

Ongoing and future work

Ongoing and future work

- Prove the general version of the 2-dimensional analogue of [Fak70].
- Study the replacement of **structure** by **property** – [Her00], [Her01], [CS10] – give a clear 2-categorical proof.
- New characterizations of lax-idempotency for 2-monads.
- Generalize more results from ‘formal theory of monads’ to (co)lax algebras of a 2-monad T.

Publications and talks

Publications

- "Factorization systems and double categories." *Theory And Applications Of Categories* 41.18 (2024): 551-592.
- "Colax adjunctions and lax-idempotent pseudomonads." *Theory and Applications of Categories*, 44.7 (2025): 227-271.

Talks I

Category theory seminars:

- 2.2.2022 **Category theory seminar**, Johns Hopkins University, Baltimore, MD: Codescent objects and lax coherence
- 9.6.2022 **Algebra seminar**, Masaryk University: On the computation of codescent objects
- 18.5.2023 **Algebra seminar**, Masaryk University: Factorization systems and double categories
- 4.10.2023 **Algebra seminar**, Masaryk University: Quasi-limits and lax flexibility

Talks II

In-person workshops and conferences:

- 6.7.2023 **Category theory 2023**, Louvain-la-Neuve, Belgium: Factorization systems as double categories
- 19.3.2024 **PTSPC**, Talinn, Estonia: Turning lax monoidal categories into strict ones
- 27.6.2024 **Category theory 2024**, Santiago de Compostela, Spain: Lax adjunctions and lax-idempotent pseudomonads
- 15.11.2024 **PSSL 109**, Leiden, Netherlands: Some characterizations of lax idempotency for pseudomonads

Talks III

Online workshops and conferences:

- 28.10.2023 **Octoberfest 2023**: Quasi-limits and lax flexibility
- 21.10.2024 **Second virtual workshop on double categories**: Double categories versus factorization systems

Bibliography I

[BKP89] Robert Blackwell, G. M. Kelly, and A. John Power. "Two-dimensional monad theory". In: *Journal of pure and applied algebra* 59.1 (1989), pp. 1–41.

[Bun74] Marta Bunge. "Coherent extensions and relational algebras". In: *Transactions of the American Mathematical Society* 197 (1974), pp. 355–390.

[CS10] Geoffrey S.H. Cruttwell and Michael A. Shulman. "A Unified Framework For Generalized Multicategories". In: *Theory and Applications of Categories* 24.21 (2010), pp. 580–655.

[DPP06] R.J. MacG. Dawson, R. Paré, and D.A. Pronk. "Paths in double categories". In: *Theory Appl. Categ* 16 (2006), pp. 460–521.

Bibliography II

[Fak70] Sabah Fakir. "Monade idempotente associée à une monade". In: *C. R. Acad. Sci. Paris Ser. A-B* 270 (1970), A99–A101.

[Gar18] Richard Garner. *Laxification, comment by Richard Garner.* Accessed: 2024-09-30. 2018. URL: `https://golem.ph.utexas.edu/category/2018/05/laxification.html#c054053`.

[Gra74] John Gray. "Formal Category Theory: Adjointness For 2-categories.". In: (1974).

[Her00] Claudio Hermida. "Representable multicategories". In: *Advances in Mathematics* 151.2 (2000), pp. 164–225.

Bibliography III

[Her01] Claudio Hermida. “From coherent structures to universal properties”. In: *Journal of Pure and Applied Algebra* 165.1 (2001), pp. 7–61.

[Koc95] Anders Kock. “Monads for which structures are adjoint to units”. In: *Journal of Pure and Applied Algebra* 104.1 (1995), pp. 41–59.

[Lac02] Stephen Lack. “Codescent objects and coherence”. In: *Journal of Pure and Applied Algebra* 175.1-3 (2002), pp. 223–241.

[Lac05] Stephen Lack. “Limits for lax morphisms”. In: *Applied Categorical Structures* 13 (2005), pp. 189–203.

[Lac14] Stephen Lack. “Morita contexts as lax functors”. In: *Applied Categorical Structures* 22.2 (2014), pp. 311–330.

Bibliography IV

[LR12] Stephen Lack and Jiří Rosický. "Enriched weakness". In: *Journal of Pure and Applied Algebra* 216.8-9 (2012), pp. 1807–1822.

[LS12] Stephen Lack and Michael A. Shulman. "Enhanced 2-categories and limits for lax morphisms". In: *Advances in Mathematics* 229.1 (2012), pp. 294–356.

[Mil03] Stefan Milius. "On colimits in categories of relations". In: *Applied Categorical Structures* 11 (2003), pp. 287–312.

[MW12] Francisco Marmolejo and R. J. Wood. "Kan extensions and lax idempotent pseudomonads". In: *Theory and Applications of Categories* 26.1 (2012), pp. 1–29.

Bibliography V

[Ště20] Miloslav Štěpán. "2-kategorie [online]". Supervisor: John Bourke. Diplomová práce. Masarykova univerzita, Přírodovědecká fakulta, Brno, 2020. URL: `https://is.muni.cz/th/ahwdt/`.

[Web15] Mark Weber. "Internal Algebra Classifiers As Codescent Objects Of Crossed Internal Categories". In: *Theory & Applications of Categories* 30.50 (2015).

[Zöb76] Volker Zöberlein. "Doctrines on 2-categories". In: *Mathematische Zeitschrift* 148 (1976), pp. 267–279.

Thank You for Your Attention!

MASARYK
UNIVERSITY