TURNING LAX ALGEBRAS INTO STRICT ONES USING KLEISLI OBJECTS

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m³dm

Plan:

- COLAX ALGEBRAS
- TURNING THEM STRICT
- COCODOCO



2-MONADS

LET $\mathcal{K}$ BE A 2-CATEGORY.

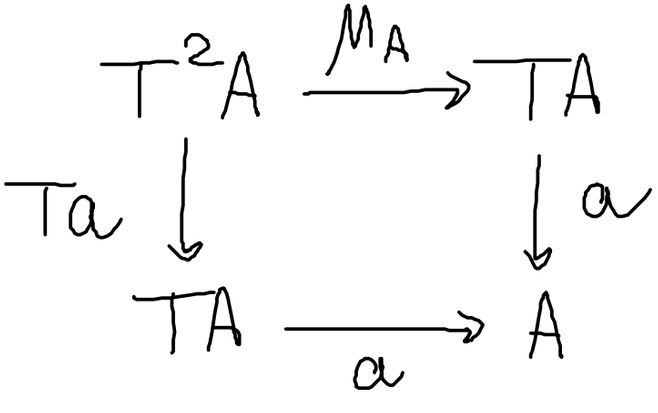
2-MONAD: $\begin{cases} T : \mathcal{K} \to \mathcal{K} \\ M : T^2 \Rightarrow T \\ \eta : 1_T \Rightarrow T \end{cases}$

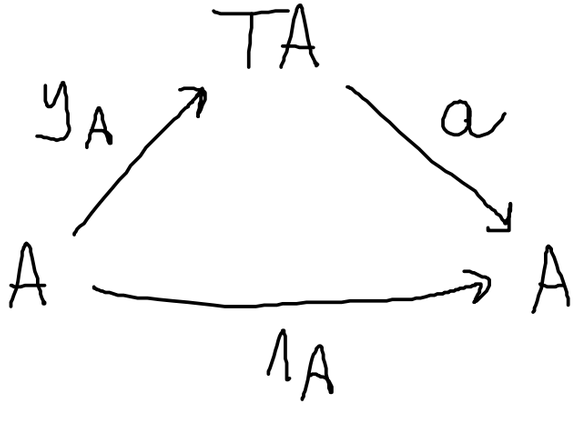
SATISFYING $M \circ TM = M \circ MT$

$M \circ T\eta = 1_T = M \circ \eta T$

CAN CONSIDER STRICT T-ALGEBRAS:

(A, a) S.T.





2-MONADS

LET $\mathcal{R}$ BE A 2-CATEGORY.

2-MONAD: $\begin{cases} T : \mathcal{R} \longrightarrow \mathcal{R} \\ M : T^2 \Rightarrow T \\ \eta : 1_T \Rightarrow T \end{cases}$

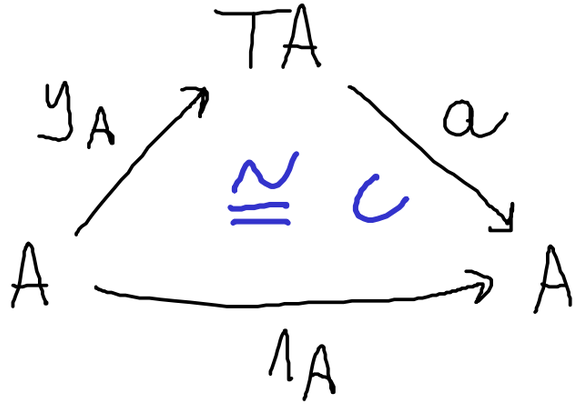
SATISFYING $M \circ TM = M \circ MT$

$M \circ T\eta = 1_T = M \circ \eta T$

CAN CONSIDER PSEUDO T-ALGEBRAS:

(A, a, η, C)

$$\begin{array}{ccc} T^2A & \xrightarrow{M_A} & TA \\ {\scriptstyle Ta}\downarrow & \cong \eta & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array}$$



2-MONADS

LET $\mathcal{K}$ BE A 2-CATEGORY.

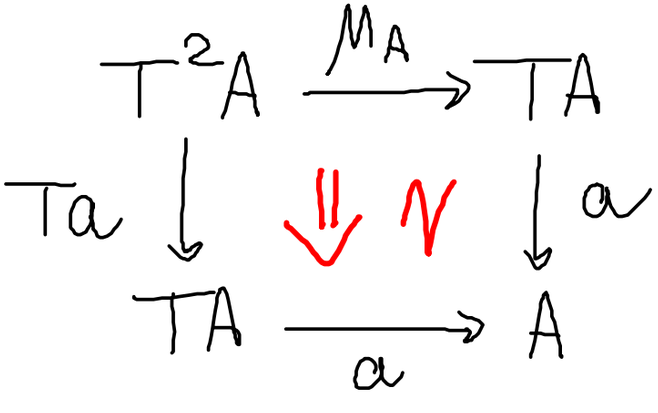
2-MONAD: $\begin{cases} T : \mathcal{K} \to \mathcal{K} \\ M : T^2 \Rightarrow T \\ \eta : 1_T \Rightarrow T \end{cases}$

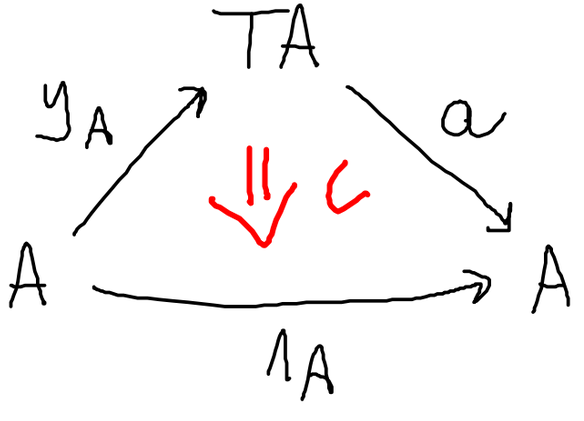
SATISFYING $M \circ TM = M \circ MT$

$M \circ T\eta = 1_T = M \circ \eta T$

CAN CONSIDER COLAX T-ALGEBRAS:

(A, a, ν, c)





ALGEBRAS

2-MONAD T ON $\mathbf{Cat}$		COLAX T-ALGEBRA
$T = 1_{\mathbf{Cat}}$	...	COMONAD $(A, q, \delta, \varepsilon)$
$T = *$	...	CATEGORY WITH INITIAL OBJECT
$A \mapsto \coprod_{n \geq 0} A^n$	...	COLAX MONOIDAL CATEGORY $(A, (\otimes^n)_{n \geq 0}, N, C)$

n-ARY

TURNING COLAX INTO STRICT

KLEISLI CATEGORY

$$\mathrm{Cat} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{CoMonad}(\mathrm{Cat})$$

[Street 72]

„J-GRADED KLEISLI CAT"

$$[J, \mathrm{Cat}] \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{ColaxFun}[J, \mathrm{Cat}]$$

[Street 72B]

FREE STRICT MONOIDAL CATEGORY
ON THE UNDERLYING MULTICATEGORY

$$\mathrm{StrictMonCat} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{ColaxMonCat}$$

$\boxed{\text{COR 6.2.8}}$

THM 4.4.3 LET T BE A 2-MONAD ON $\mathbf{Cat}$ ARISING FROM A CARTESIAN MONAD ON $\mathbf{Set}$.

THEN $\mathbf{Strict}\text{-}T\text{-}\mathbf{Alg}_s \xrightarrow{(-)'} \mathbf{Colax}\text{-}T\text{-}\mathbf{Alg}_c$

HOW ?

- TAKE A COLAX ALGEBRA $\mathbb{A}$,
- FORM A DIAGRAM $\mathrm{Res}(\mathbb{A})$
- COMPUTE ITS CODESCENT OBJECT $\mathbb{A}'$

[Lack2002], [Weber15]

THE DIAGRAM Res(A) IS AN INSTANCE OF A CODOMAIN COLAX DOMAIN CODISCRETE CATEGORY.

$\mathbb{X} :=$

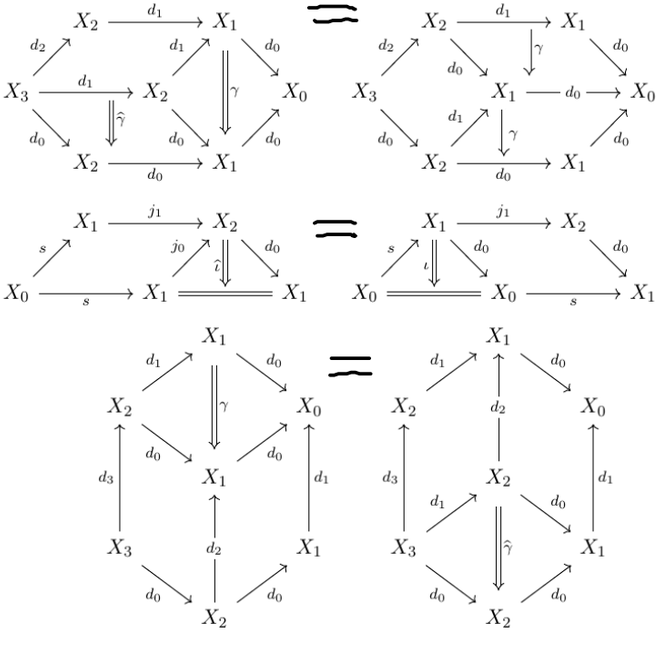
$$X_3 \;\underset{d_0^{\bullet\bullet}}{\overset{d_3^{\bullet\bullet}}{\underset{d_1^{\bullet\bullet}}{\overset{d_2^{\bullet\bullet}}{\rightrightarrows}}}}\; X_2 \;\underset{j_0}{\overset{d_2^{\bullet}}{\underset{d_0^{\bullet}}{\overset{j_1}{\underset{d_1^{\bullet}}{\rightleftarrows}}}}}\; X_1 \;\underset{d_0}{\overset{d_1}{\rightleftarrows}}\; X_0$$

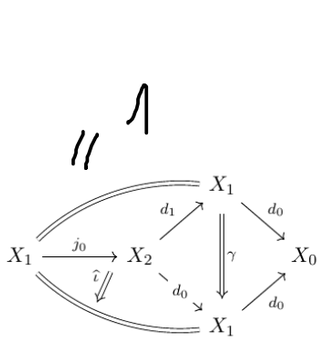
$$\begin{aligned}
\gamma &: d_0 d_1^{\bullet} \Rightarrow d_0 d_0^{\bullet},\\
\hat{\gamma} &: d_0^{\bullet} d_1^{\bullet\bullet} \Rightarrow d_0^{\bullet} d_0^{\bullet\bullet},\\
\iota &: d_0 s \Rightarrow 1_{X_0},\\
\hat{\iota} &: d_0^{\bullet} j_0 \Rightarrow 1_{X_1}.
\end{aligned}$$

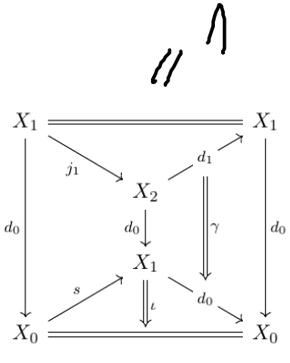
$$\begin{aligned}
d_1 d_1^{\bullet} &= d_1 d_2^{\bullet}, & d_0 d_2^{\bullet} &= d_1 d_0^{\bullet},\\
d_1^{\bullet} d_1^{\bullet\bullet} &= d_1^{\bullet} d_2^{\bullet\bullet}, & s d_1 &= d_2^{\bullet} j_0,\\
d_2^{\bullet} d_2^{\bullet\bullet} &= d_2^{\bullet} d_3^{\bullet\bullet}, & d_1^{\bullet} d_3^{\bullet\bullet} &= d_2^{\bullet} d_1^{\bullet\bullet},\\
& & d_0^{\bullet} d_2^{\bullet\bullet} &= d_1^{\bullet} d_0^{\bullet\bullet},\\
& & d_0^{\bullet} d_3^{\bullet\bullet} &= d_2^{\bullet} d_0^{\bullet\bullet}.
\end{aligned}$$

$$\begin{aligned}
d_1 s &= 1_{X_0},\\
d_1^{\bullet} j_0 &= 1_{X_1},\\
d_2^{\bullet} j_1 &= 1_{X_1},\\
d_2 j_1 &= 1_{X_1},
\end{aligned}$$

$$\begin{aligned}
s d_0 &= d_0^{\bullet} j_1,\\
j_0 s &= j_1 s.
\end{aligned}$$







& d_1 IS A DISCRETE OPFIBRATION

(& CERTAIN PULLBACK CONDITIONS)

THE ABOVE IS JUST A MONAD IN

$Kl(\underbrace{\text{Span}}_{\text{BICATEGORY OF SPANS}})$

$\underbrace{\phantom{Kl(\text{Span})}}_{\text{FREELY ADD KLEISLI OBJECTS OF MONADS}}$

EQUIVALENTLY, AN OBJECT OF $Kl(Kl(\text{Span}))$

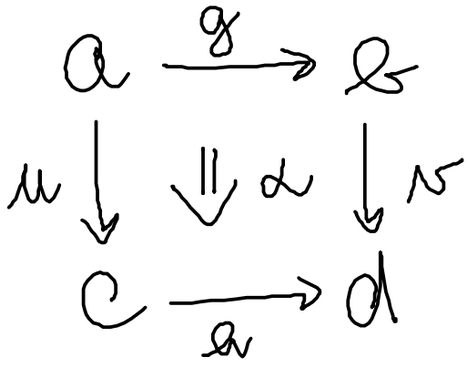
WHAT IS IT IN ELEMENTARY TERMS ?

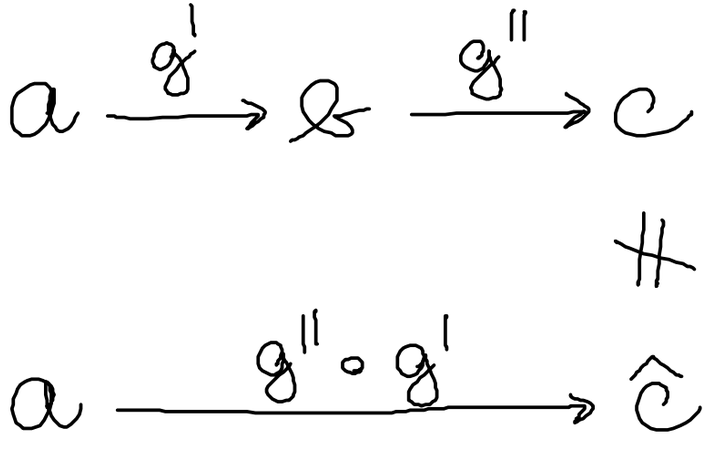
CODOMAIN COLAX DOMAIN CODISCRETE

⇝ COCODOCO

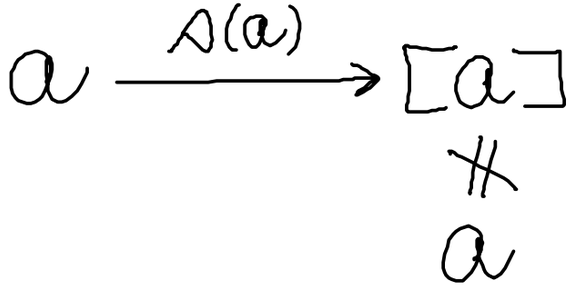
WHAT IS IT IN ELEMENTARY TERMS ?

X IS LIKE A DOUBLE CATEGORY EXCEPT:





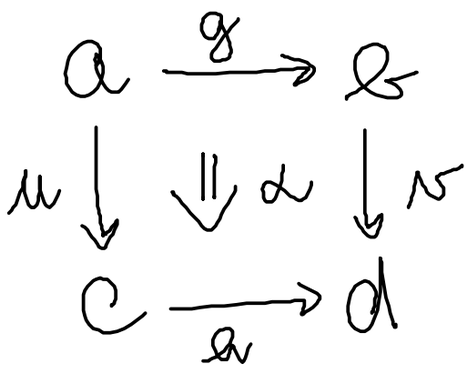
HORIZONTAL COMPOSITION

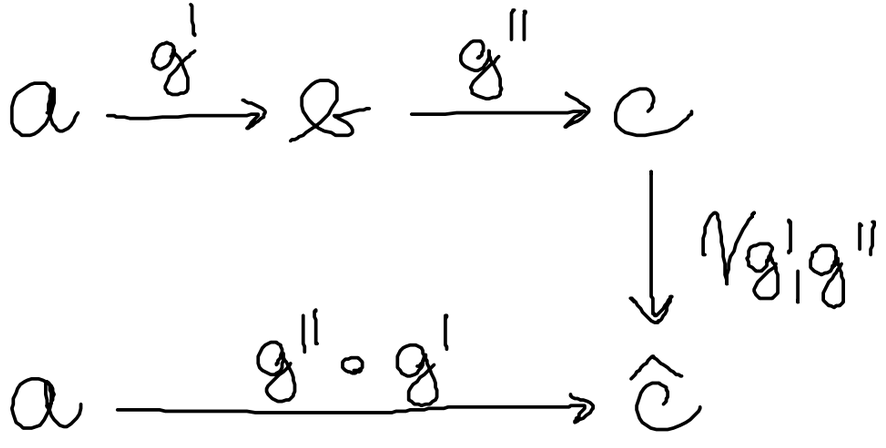


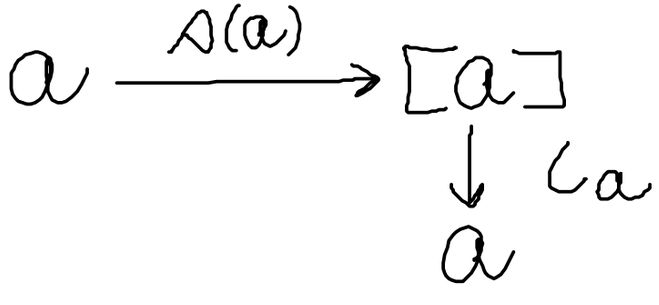
HORIZONTAL IDENTITY

WHAT IS IT IN ELEMENTARY TERMS ?

X IS LIKE A DOUBLE CATEGORY EXCEPT:







EXAMPLES:

- PARTIAL EVALUATIONS $(Set, \times, 1)$

$$(A, B, C) \longrightarrow (A, B \times C) \longrightarrow (A \times (B \times C))$$

$$(A, B, C) \longrightarrow (A \times B \times C)$$

- FOR A COLAX FUNCTOR $(P, \eta, c) : \mathbb{J} \to Cat$

$$(j', x') \xrightarrow{\alpha} (j'', x'') \qquad \alpha : j' \to j''$$

$$\text{S.T. } P(\alpha)(x') = x''$$

- MODELS OF MLTT

$$\Gamma.A.B \longrightarrow \Gamma.A \longrightarrow \Gamma$$

INTRODUCTION $\downarrow$

$$\Gamma.\Sigma(A, B) \longrightarrow \Gamma$$

LITERATURE:

- MY THESIS
- MY TALK AT PTSAC 24

  ⇝ milostepan.github.io

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  Cahiers de topologie et géométrie différentielle 13.3 (1972): 217-264.

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  Weber, Mark. "Internal algebra classifiers as codescent
  objects of crossed internal categories."
  arXiv preprint arXiv:1503.07585 (2015).

THANK YOU!

