adjointable maps for graphs

MILOSLAV ŠTĚPÁN

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JOINT W/

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Plan:

1. MOTIVATION
2. ELEMENTARY EXAMPLES
3. OBSERVATIONS
4. CYCLES & PATHS
5. OPEN QUESTIONS

DEFINITION)

GIVEN GRAPHS $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$
A FUNCTION $f : V_1 \to V_2$ IS **ADJOINTABLE**
IF THERE IS $g : V_2 \to V_1$
SUCH THAT:

$$f(v_1)v_2 \in E_2 \iff v_1 g(v_2) \in E_1$$

FOR ALL $v_1 \in V_1$, $v_2 \in V_2$

I DENOTE $\boxed{f : G_1 \to G_2}$

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LET $(\mathcal{H}, \langle -, - \rangle), (\mathcal{K}, \langle -, - \rangle)$ BE HILBERT SPACES.
TAKE A PAIR OF LINEAR MAPS

$$\mathcal{H} \underset{f}{\overset{g}{\leftrightarrows}} \mathcal{K}$$

SATISFYING $\langle fu, v \rangle = \langle u, gv \rangle$
FOR ALL $u \in \mathcal{H}, v \in \mathcal{K}$

CONSIDER GRAPHS $(\mathcal{H}, \perp), (\mathcal{K}, \perp)$
$\rightsquigarrow$ ADJOINTABLE MAPS

$$(\mathcal{H}, \perp) \underset{f}{\overset{g}{\leftrightarrows}} (\mathcal{K}, \perp)$$

[PV25], [FF13]

GIVEN GRAPHS $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$
A FUNCTION $f : V_1 \to V_2$ IS ADJOINTABLE
IF THERE IS $g : V_2 \to V_1$
SUCH THAT:

$$f(v_1)v_2 \in E_2 \iff v_1 g(v_2) \in E_1$$

- f, g BIJECTIVE, THE PAIR (f, g^{-1}) IS AN ISOTOPY OF [Zelinka72]

  ALSO CALLED A TF-AUTOMORPHISM [LMS11]

- f BIJECTIVE SELF-ADJOINT IN [CHIK11]

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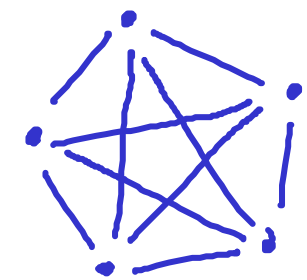
DENOTE BY $\boxed{K_n}$ THE COMPLETE GRAPH ON VERTICES $\{0, \dots, n-1\}$

K_5 

PROPOSITION $K_n \underset{f}{\overset{g}{\leftrightarrows}} K_m$ ADJOINT IFF $n = m$ & $f = g^{-1}$

PROOF:

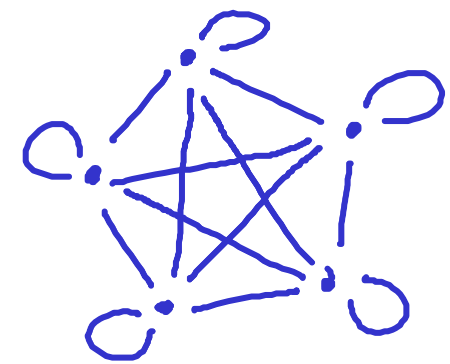
$$f(x) \neq y \iff x \neq g(y)$$

I.E. $$f(x) = y \iff x = g(y)$$

PUT $x := g(y)$, GET $f(g(y)) = y$. $\square$

DENOTE BY $\boxed{K^\circ_n}$ THE COMPLETE GRAPH WITH LOOPS

K°_5



$\boxed{\text{PROPOSITION}}$ ANY TWO FUNCTIONS $K^\circ_n \underset{f}{\overset{g}{\rightleftarrows}} K^\circ_m$ ARE ADJOINT.

GIVEN GRAPH $G=(V,E)$ AND $v \in V$,

DENOTE $\boxed{N(v)} := \{ x \in V \mid vx \in E \}$

$\boxed{\text{PROPOSITION}}$ $f : (V_1, E_1) \longrightarrow (V_2, E_2)$ IS ADJOINTABLE IFF FOR ALL $y \in V_2$ THERE IS $x_y \in V_1$ S.T. $f^{-1}(N(y)) = N(x_y)$

"PREIMAGE OF A NEIGHBORHOOD IS A NEIGHBORHOOD"

$\boxed{\text{EXAMPLE}}$ THE UNIQUE FUNCTION $f : G \longrightarrow K_1$ IS ADJOINTABLE IFF G ADMITS AN ISOLATED VERTEX

$f^{-1}(N(*)) = \emptyset \overset{?}{=} N(x)$ FOR $x \in V$

DEFINITION) A PERFECT CODE IN $G=(V,E)$ IS A SUBSET $S \subseteq V$ S.T.

$\forall x \in V \ \exists! \, a \in S$ SUCH THAT $x \in N(a) \cup \{a\}$

PROPOSITION LET G BE A GRAPH.

PERFECT CODES IN G $\equiv$ ADJOINTABLE MAPS $K_n \to \overline{G}$

COMPLEMENT OF G

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DEFINITION) A **GRAPH HOMOMORPHISM**

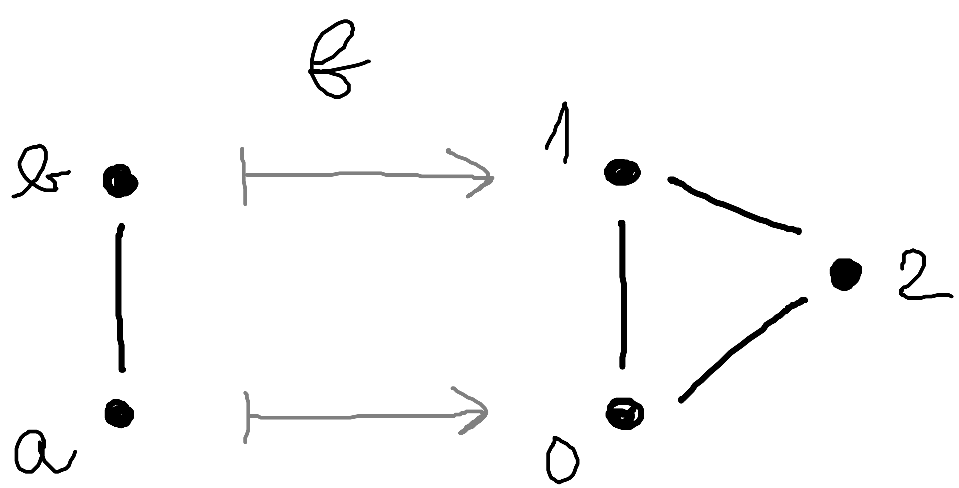
$$f : (V_1, E_1) \longrightarrow (V_2, E_2)$$

A FUNCTION $f : V_1 \to V_2$ SATISFYING

$$v_1 v_2 \in E_1 \Rightarrow f(v_1) f(v_2) \in E_2$$

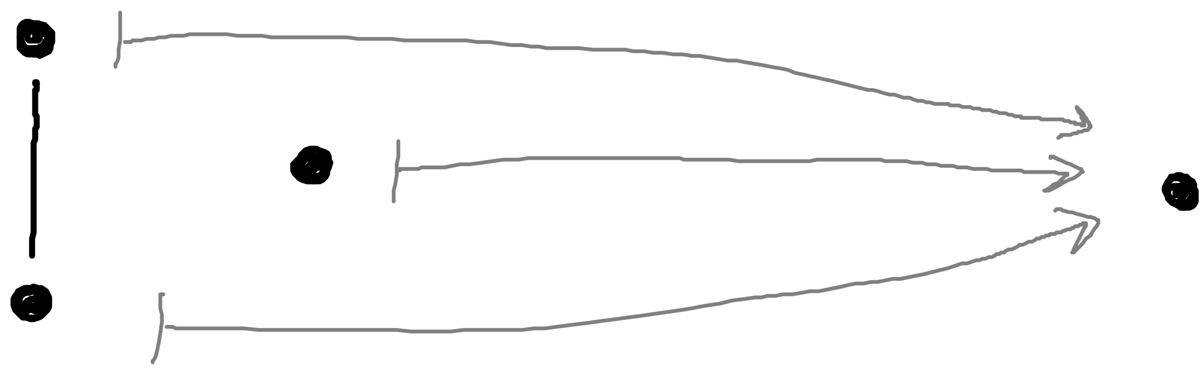
GRAPH HOMOMORPHISM

$\not\Rightarrow$ ADJOINTABLE



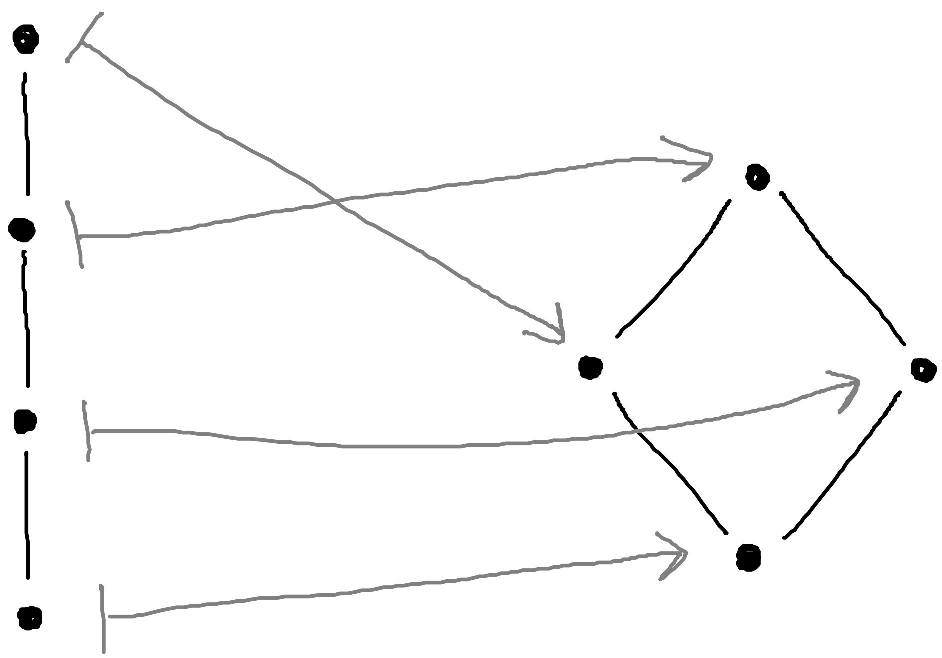
$$f^{-1}(N(2)) = f^{-1}(\{0,1\})$$
$$= \{a, b\}$$
$$\neq N(a)$$
$$\neq N(b)$$

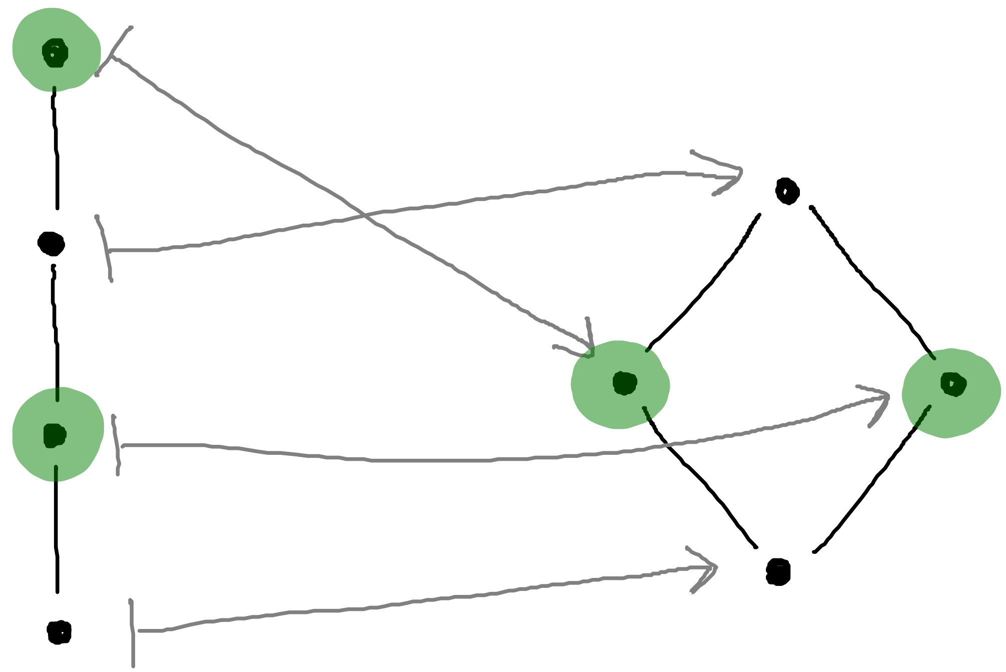
ADJOINTABLE $\not\Rightarrow$ GRAPH HOMOMORPHISM

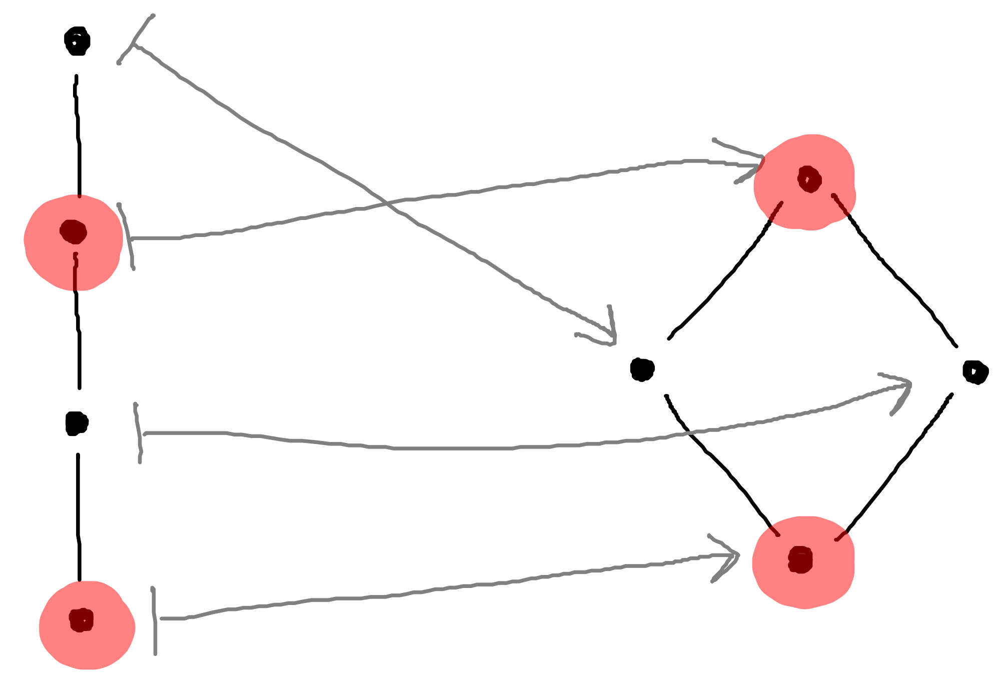


$f : G \to G'$ ADJOINTABLE BIJECTION

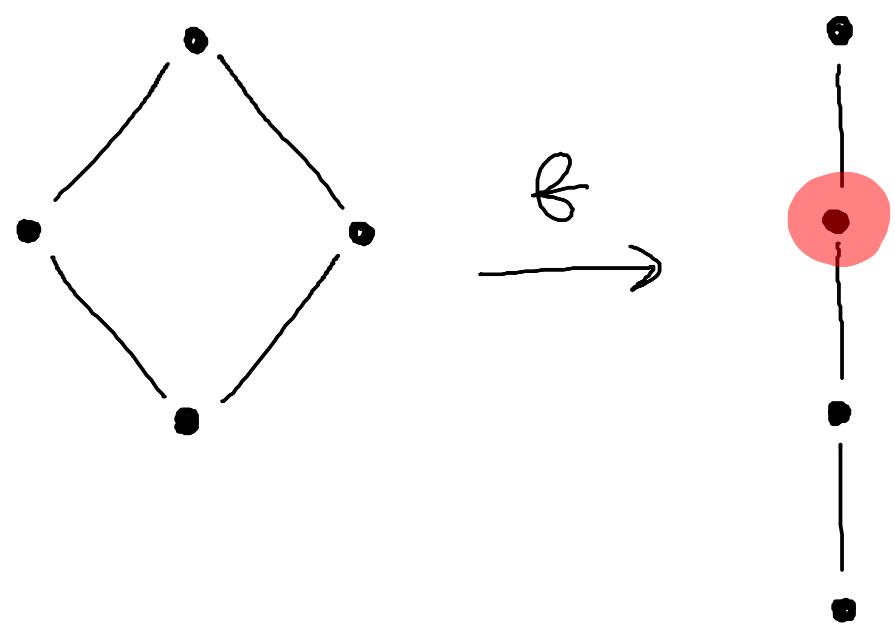
$\not\Longrightarrow$ f^{-1} ADJOINTABLE



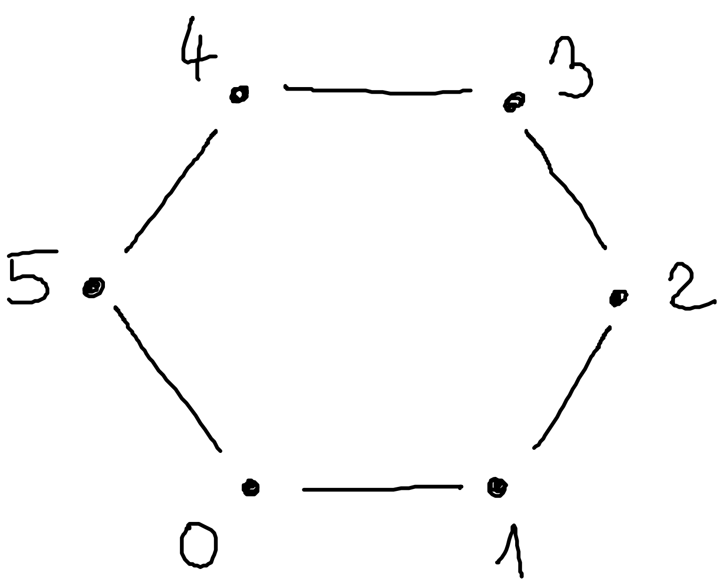


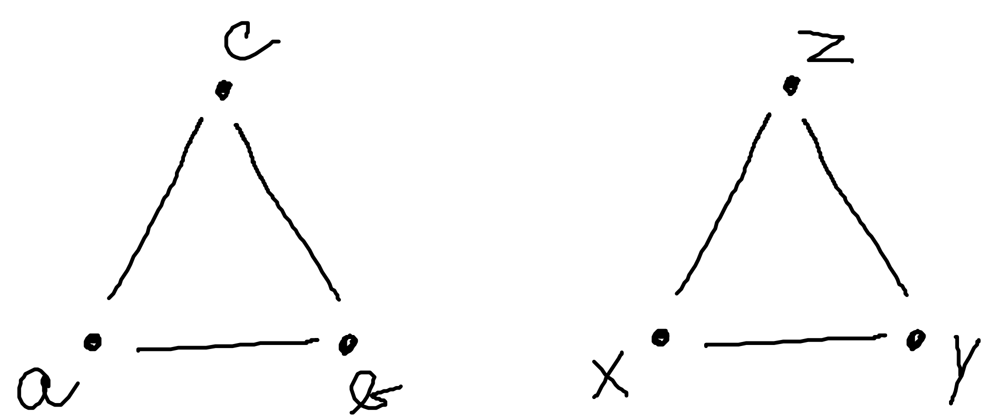


BUT THERE IS NO ADJOINTABLE BIJECTION



CAN THERE BE ADJOINTABLE BIJECTIONS $G \underset{f}{\overset{g}{\leftrightarrows}} G'$ WHEN $G \not\cong G'$? YES:





ALSO SEE [LMS11]

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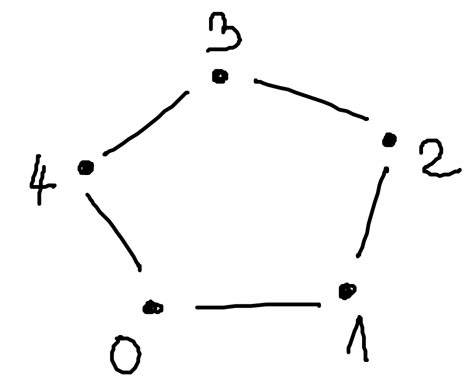
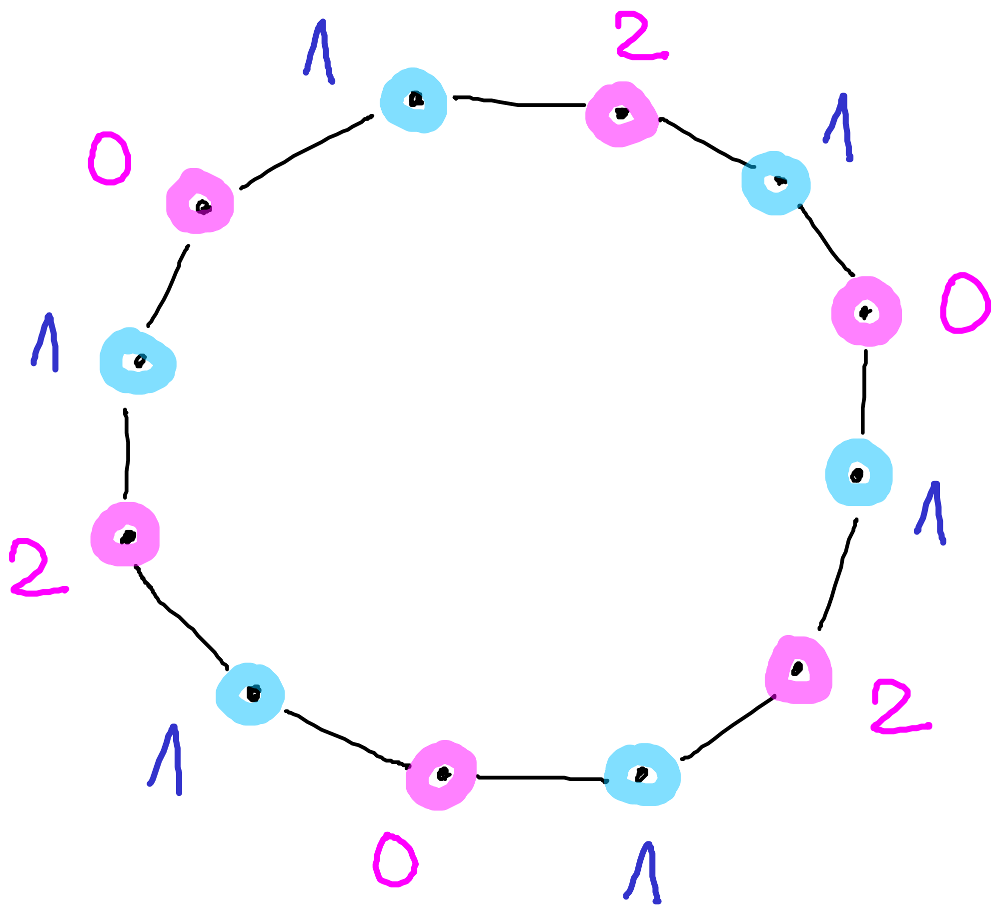
DENOTE BY C_n THE CYCLE GRAPH ON VERTICES $\{0, 1, \dots, n-1\}$



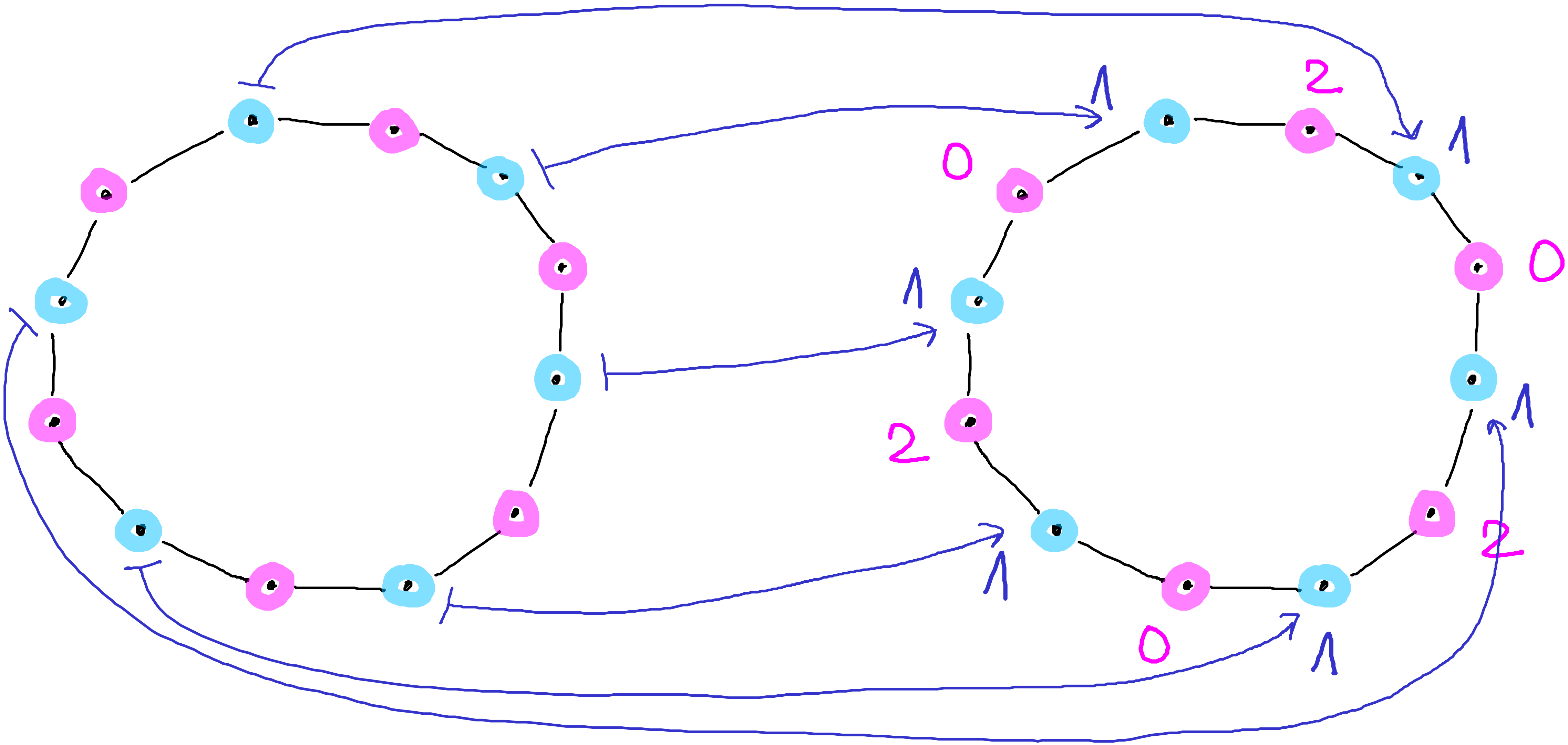
THEOREM LET $n \geq 5$.
THE NUMBER OF ADJOINTABLE MAPS $C_n \to C_n$ IS:

$$\begin{cases} 2n & \text{WHEN } n \text{ IS ODD} \\ 2n^2 & \text{WHEN } n \equiv 2 \text{ MOD } 4,\ n \geq 6 \\ 2n^2 + 16n\left(\frac{n}{4}\right)! + 16\left(\frac{n}{2}\right)! & \text{WHEN } n \equiv 0 \text{ MOD } 4,\ n \geq 8 \end{cases}$$

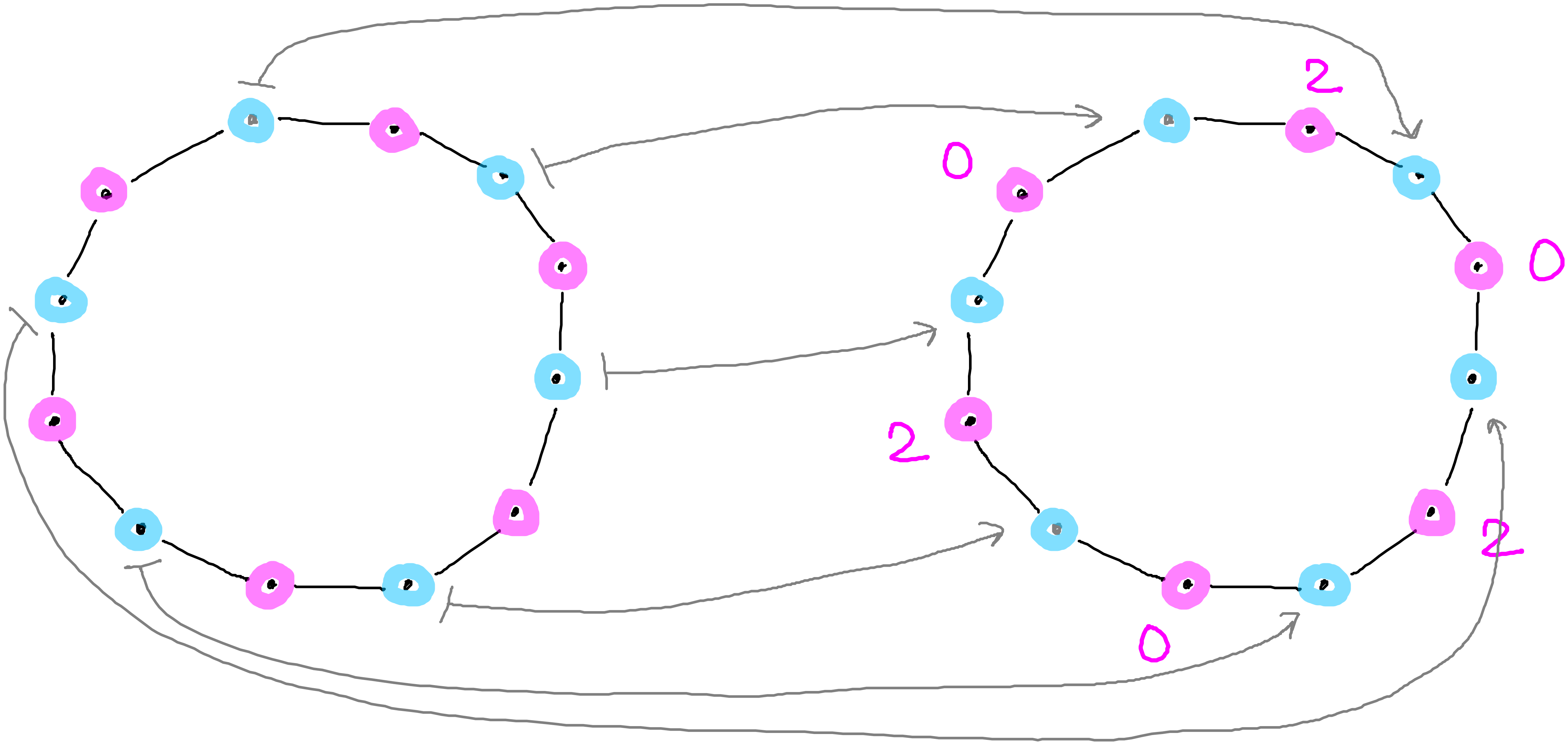
INTUITION: $n = 12$



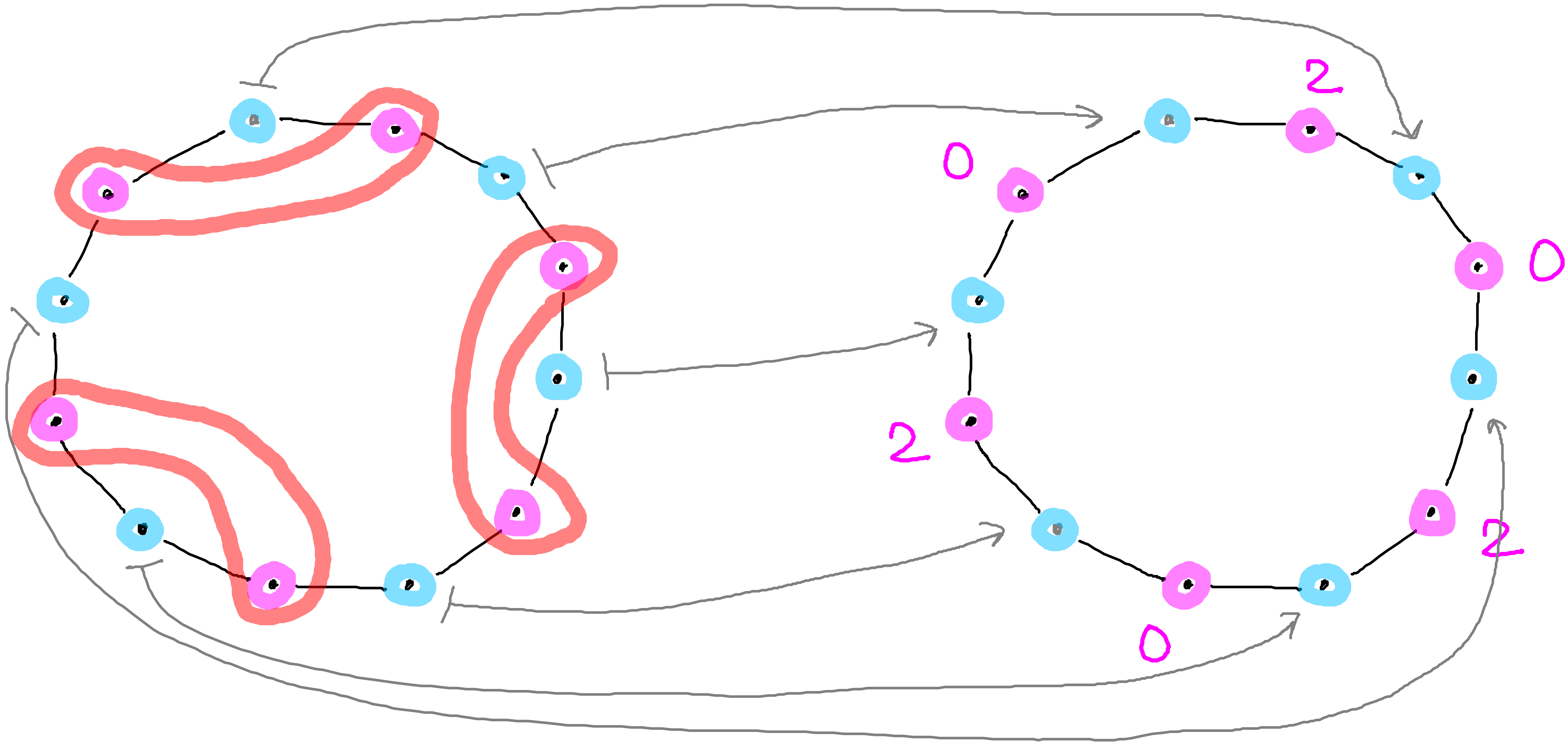
- TAKE ODD AND EVEN VERTICES.
- EACH COLOR HAS PATTERN 111... OR 2020...



1
2
1
0
0
1
2
1
2
1
0
1



2
0
0
2
2
0


2
0
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2
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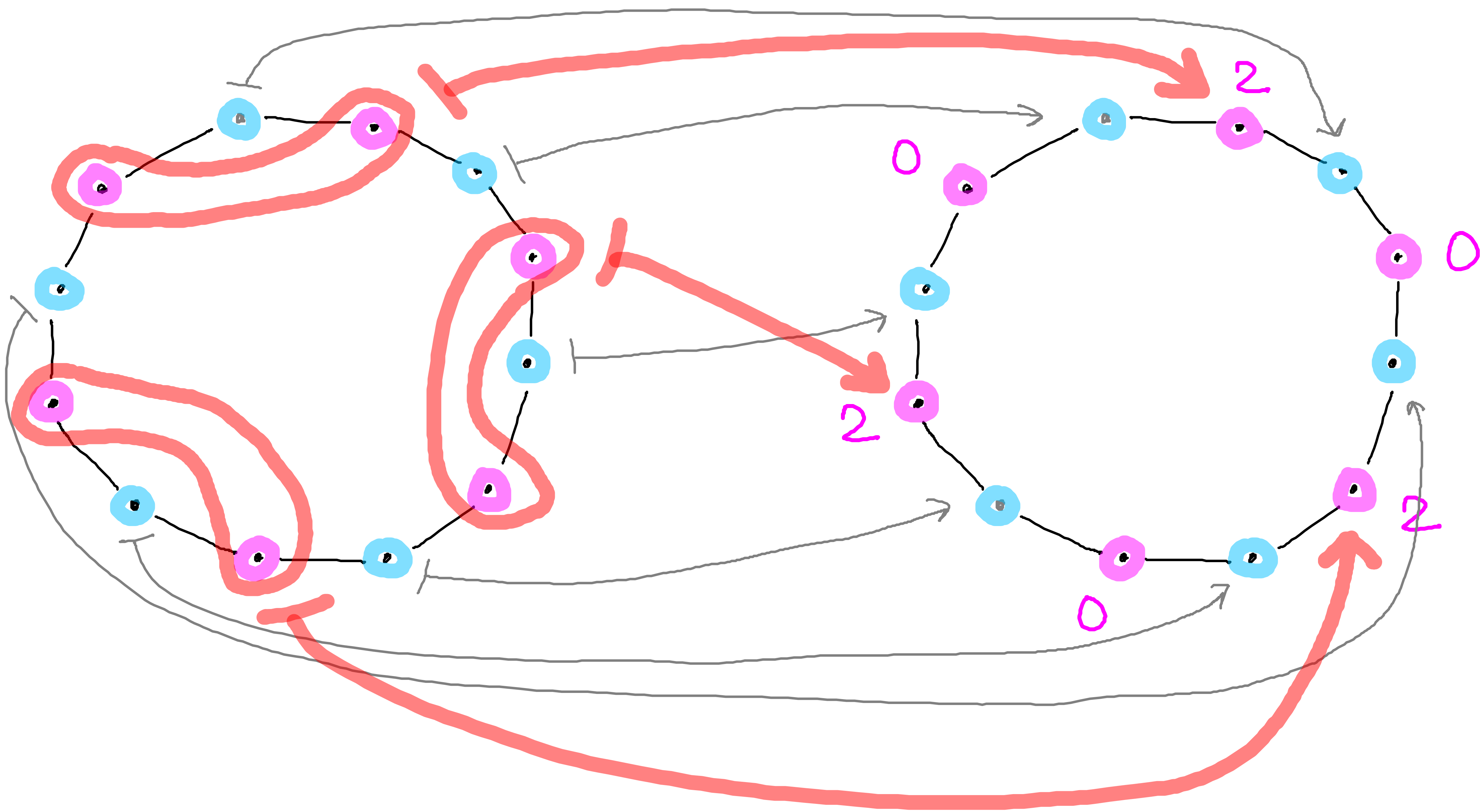


2
0
0
2
2
0

$\boxed{\text{THEOREM}}$ [MK08]
THE NUMBER OF GRAPH HOMOMORPHISMS $C_n \to C_n$

$$\text{IS} \begin{cases} 4n + 2 & \text{WHEN } n \text{ IS ODD} \\ 2n\left(1 + \frac{1}{2}\binom{2n}{n}\right) & \text{WHEN } n \text{ IS EVEN} \end{cases}$$

$\rightsquigarrow$ $n \equiv 0 \mod 4$ ADJOINTABLES GROW MUCH FASTER

DENOTE BY $\boxed{P}_n$ THE PATH ON VERTICES $\{0, 1, \dots, n-1\}$

$$\underset{0}{\bullet} - \underset{1}{\bullet} - \cdots - \underset{n-2}{\bullet} - \underset{n-1}{\bullet}$$

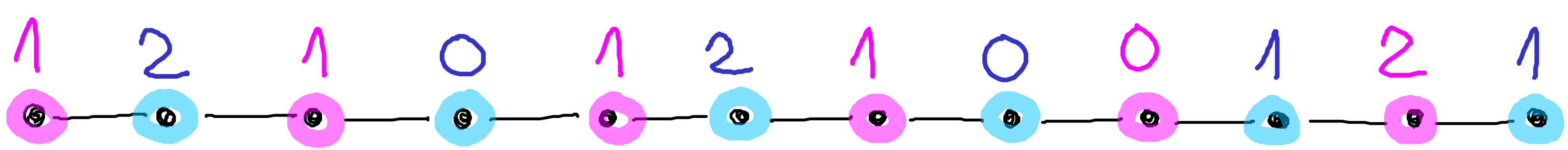
CALCULATING $b : P_n \to P_n \ldots$

$n \equiv 1$ MOD 4: ONLY PATTERN IS $111\ldots$

$n \equiv 0$ MOD 4: BLUE PATTERN $(20)^x 1^y$

PINK PATTERN $1^a (02)^b$

FOR SOME $x, y, a, b \geq 0$



OPEN PROBLEMS

- [Cameron26] SHOWS VARIOUS GRAPHS ASSOCIATED TO GROUPS ETC.
  WHAT ARE ADJOINTABLES THERE ?
- FOR WHAT CLASSES OF GRAPHS:
  (1) ADJOINTABLES $\subseteq$ GRAPH HOMOMORPHISMS
  (2) f BIJ ADJOINTABLE
  $\Rightarrow f^{-1}$ ADJOINTABLE ?
- IS THE PROBLEM OF „ADJ-ISOMORPHISM" P OR NP ?

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Thank you!