SOME CHARACTERIZATIONS OF LAX IDEMPOTENCY FOR PSEUDOMONADS

MILOSLAV ŠTĚPÁN
milosla v.stepan@mail.muni.cz

109th Peripatetic Seminar
On Sheaves and Logic

November 15, 2024

PLAN OF THE TALK

1) RECALL PSEUDOMONADS (T, m, i) & LAX IDEMPOTENCY

2) CHARACTERIZATIONS INVOLVING:

  2.1) REFLECTORS TO $i : 1_{\mathcal{K}} \Rightarrow T$

  2.2) „KZ-FICATION PROCESS"

  2.3) COLAX ADJUNCTIONS

  2.4) COLIMITS OF ARROWS

$\boxed{1}$ PSEUDOMONADS & ALGEBRAS

DEF) A PSEUDOMONAD ON A 2-CATEGORY $\mathcal{K}$ CONSISTS OF:

- PSEUDOFUNCTOR T
- PSEUDONATURAL $m : T^2 \Rightarrow T$, $i : 1 \Rightarrow T$
- INVERTIBLE MODIFICATIONS:

$$\begin{tikzcd}
T^3 \arrow[r, "mT"] \arrow[d, "Tm"'] \arrow[dr, phantom, "\cong \; \Uparrow \mu"] & T^2 \arrow[d, "m"] \\
T^2 \arrow[r, "m"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "iT"] \arrow[dr, equal] \arrow[dr, phantom, "\beta \Downarrow", near start, shift left=3] & T^2 \arrow[d, "m"] & T \arrow[l, "Ti"'] \arrow[dl, equal] \arrow[dl, phantom, "\Leftarrow \vartheta", near start, shift right=3] \\
 & T &
\end{tikzcd}$$

SUBJECT TO EQUATIONS.

WILL DENOTE BY $\boxed{(T, m, i)}$

DEF) (T, m, i) PSEUDOMONAD ON $\mathfrak{K}$.

A **COLAX T-ALGEBRA** CONSISTS OF:

- $A \in \operatorname{ob} \mathfrak{K}$ • $a : TA \to A \in \operatorname{mor} \mathfrak{K}$
- 2-CELLS:

$$\begin{tikzcd}
T^2A \arrow[r, "Ta"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\Uparrow \gamma"] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, "a"] & A \\
A \arrow[u, "i_A"] \arrow[ur, equal, "\Downarrow c"]
\end{tikzcd}$$

SUBJECT TO EQUATIONS. DENOTE BY (A, a, γ, c).

IF c INVERTIBLE ... **NORMAL COLAX ALG.**

IF γ, c INVERTIBLE ... **PSEUDO ALG.**

Examples 1/2

EXAMPLE $1_{Cat} \circlearrowright Cat$

STRICT/PSEUDO ALGEBRAS $\leftrightsquigarrow$ SMALL CATS

COLAX ALGEBRAS $\leftrightsquigarrow$ COMONADS

EXAMPLE $T \circlearrowright Cat$, $T\mathcal{A} := *$

STRICT/PSEUDO ALGEBRAS $\leftrightsquigarrow$ $*$

COLAX ALGEBRA $\leftrightsquigarrow$ CATEGORY $\mathcal{A}$ W/ AN INITIAL OBJECT

Examples 2/2

EXAMPLE $T \circlearrowright \mathrm{Cat}$, $TA = \coprod_{n \geq 0} A^n$

$\rightsquigarrow$ MONOIDAL CATEGORIES

EXAMPLE $\mathcal{P} \circlearrowright \mathrm{CAT}$, $\mathcal{P}\mathcal{C}$ = SMALL PRESHEAVES $\mathcal{C}^{op} \longrightarrow \mathrm{Set}$

PSEUDO ALGEBRAS $\leftrightsquigarrow$ COCOMPLETE CATS

EXAMPLE $\mathcal{J}$ SMALL 2-CAT'RY, $\iota : \mathrm{ob}\,\mathcal{J} \hookrightarrow \mathcal{J}$

$T \circlearrowright [\mathrm{ob}\,\mathcal{J}, \mathrm{Cat}]$, $TX = (\mathrm{Lan}_\iota X) \circ \iota$

$\rightsquigarrow$ COLAX/PSEUDO/2-FUNCTORS $\mathcal{J} \longrightarrow \mathrm{Cat}$

RECALL:

PROP THE FOLLOWING ARE EQUIVALENT FOR A PSEUDOMONAD (T, m, i) ON $\mathcal{K}$:

- $m \dashv iT$ IN $\mathrm{Hom}[\mathcal{K}, \mathcal{K}]$
- $Ti \dashv m$ IN $\mathrm{Hom}[\mathcal{K}, \mathcal{K}]$
- THERE IS A MODIFICATION

$$\begin{tikzcd} T \arrow[r, bend left, "Ti"] \arrow[r, bend right, "iT"'] \arrow[r, phantom, "\Downarrow \lambda"] & T^2 \end{tikzcd}$$

SATISFYING (...)

IN THIS CASE SAY (T, m, i) IS **LAX-IDEMPOTENT** (ALSO **KZ**)

- SATISFY VARIOUS PROPERTIES 💬

$\boxed{2.1}$ CHARACTERIZATION INVOLVING REFLECTORS TO $i : 1_{\mathcal{X}} \Rightarrow T$

$\boxed{2.1}$

LET (T, m, i) PSEUDOMONAD ON $\mathcal{K}$.

DEFINE Adj(i) – A 2-CATEGORY W/

OB: PAIRS $\left(A, (\varepsilon, \eta) : \begin{tikzcd} A \arrow[r, bend right, "i_A"'] \arrow[r, phantom, "\perp"] & TA \arrow[l, bend right, "u"'] \end{tikzcd}\right)$

HOMS: $\mathrm{Adj}(i)\big((A,(\varepsilon,\eta)),(B,(\tilde{\varepsilon},\tilde{\eta}))\big) := \mathcal{K}(A,B)$

FACT 1:

$(c, \sigma) : \begin{tikzcd} A \arrow[r, bend right, "i_A"'] \arrow[r, phantom, "\perp"] & TA \arrow[l, bend right, "a"'] \end{tikzcd} \Longrightarrow$ $\exists!\,\gamma$ S.T. (A, a, γ, c) IS COLAX T-ALGEBRA

FACT 2:

$$\begin{tikzcd} (c,\sigma): A \arrow[r, bend right, "i_A"'] \arrow[r, phantom, "\perp"] & TA \arrow[l, bend right, "a"'] , & (c',\sigma'): B \arrow[r, bend right, "i_B"'] \arrow[r, phantom, "\perp"] & TB \arrow[l, bend right, "b"'] \end{tikzcd}$$

$\forall f : A \to B$ $\exists!$ 2-CELL $\bar{f}$

S.T. $(f, \bar{f}) : (A, a, \eta, c) \longrightarrow (B, b, \eta', c')$

LAX T-ALGEBRA MORPHISM

FACT 3: GIVEN ANOTHER $g : A \to B$,

$\forall$ 2-CELL $\alpha : f \Rightarrow g$

IS ALGEBRA 2-CELL $\alpha : (f, \bar{f}) \Rightarrow (g, \bar{g})$

$\rightsquigarrow$ GIVES A 2-FULLY FAITHFUL 2-FUNCTOR

$\Phi : \mathrm{Adj}(i) \hookrightarrow \mathrm{NColax\text{-}T\text{-}Alg}_\ell$ ← LAX MORPHISMS

NORMAL COLAX ALGEBRAS

DENOTE $\boxed{\mathrm{Refl}(i)} \subseteq \mathrm{Adj}(i)$ ... SPANNED BY REFLECTIONS

THM THE FOLLOWING ARE EQUIVALENT FOR A PSEUDOMONAD (T, m, i):

- T IS LAX-IDEMPOTENT
- $\Phi|_{\mathrm{Refl}(i)} : \mathrm{Refl}(i) \hookrightarrow \text{NColax-}T\text{-Alg}_e$ IS AN ISOMORPHISM OF CATEGORIES.

MOREOVER, IN THIS CASE:

$$\text{NColax-}T\text{-Alg}_e = \underset{\substack{\uparrow \\ \text{PSEUDO} \\ \text{ALGEBRAS}}}{\text{Ps}}\text{-}T\text{-Alg}_e$$

2.2 CHARACTERIZATION INVOLVING „KZ-FICATION PROCESS"

2.2

RECALL: 1-MONAD (D, m, η) ON CATEGORY $\mathcal{C}$ IS IDEMPOTENT IF m_A ISO $\forall A \in \mathcal{C}$

THIS HAPPENS IFF $\eta_{TA} = T\eta_A$ $\forall A \in \mathcal{C}$.

* * *

W/ PULLBACKS

GIVEN MONAD (T, m, η) ON $\mathcal{C}$, CAN DO:

$$\begin{tikzcd} DA \arrow[r, dashed, "e"] & TA \arrow[r, shift left, "\eta_{TA}"] \arrow[r, shift right, "T\eta_A"'] & T^2A \end{tikzcd}$$

⇝ INDUCES A NEW MONAD D ON $\mathcal{C}$

⇝ CAN ITERATE THIS:

$$T \hookleftarrow D_1 \hookleftarrow D_2 \hookleftarrow \cdots$$

UNDER SUITABLE CONDITIONS THIS GIVES AN IDEMPOTENT MONAD:

- AFTER 0 STEPS: IF T ALREADY IDEMP. (SINCE $\eta_{TA} = T\eta_A$)
- AFTER 1 STEP: IF T PRESERVES COREFL. EQUALIZERS
- AFTER ∞ STEPS: IF $\mathcal{C}$ COMPL & WELL-POW YOU TAKE LIMIT OF: $T \hookleftarrow D_1 \hookleftarrow D_2 \hookleftarrow \dots$

Question?

WHAT IS A 2-DIMENSIONAL ANALOGUE?

IDEMPOTENT $\rightsquigarrow$ LAX/PSEUDO-IDEMP

EQUALIZER $\rightsquigarrow$ DESCENT OBJECT

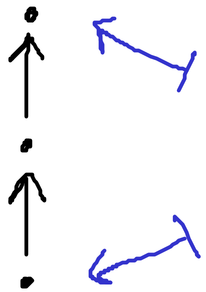
SIMPLICIAL NOTATION

DEF) $[n] := \{0 \to 1 \to \ldots \to n\} \in \mathrm{Cat}$

DEFINE SUBCATEGORY OF Cat:

$$\begin{tikzcd} \Delta_2 := & {[0]} \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_0"'] & {[1]} \arrow[l, "\sigma_0"'] \arrow[r, shift left=3, "\delta_2"] \arrow[r, "\delta_1"] \arrow[r, shift right=3, "\delta_0"'] & {[2]} \end{tikzcd}$$

HERE $\delta_j^{-1}(j) = \emptyset$ I.E.



$: \delta_1$

HAVE CANONICAL FUNCTOR $W : \Delta_2 \hookrightarrow \mathrm{Cat}$

W-WEIGHTED LIMIT OF $X : \Delta_2 \to \mathcal{X}$

— DESCENT OBJECT OF X

Example

RECALL: (T, m, y) 1-MONAD ON $\mathcal{C}$

$(A, a), (B, b)$ T-ALGEBRAS

$$\varphi \mapsto \varphi \circ a$$

$$\begin{tikzcd}
\mathcal{C}^T((A,a),(B,b)) \arrow[r, hook, "EQ"] & \mathcal{C}(A,B) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{C}(TA,B)
\end{tikzcd}$$

$$\varphi \mapsto b \circ T\varphi$$

EXAMPLE FOR A 2-MONAD (T, m, i) ON $\mathcal{K}$

$\text{Colax-}T\text{-Alg}_{\ell}((A,a,\gamma,c),(B,b))$ IS THE DESCENT

OBJECT OF

$$\begin{tikzcd}
\mathcal{K}(A,B) \arrow[r, shift left=2, "b_* \circ T(-)"] \arrow[r, shift right=2, "a^*"'] & \mathcal{K}(TA,B) \arrow[l, "i_A^*"'] \arrow[r, shift left=3, "b_* \circ T(-)"] \arrow[r, "m_A^*"] \arrow[r, shift right=3, "Ta^*"'] & \mathcal{K}(T^2A,B)
\end{tikzcd}$$

LET (T, m, i) BE A 2-MONAD ON $\mathfrak{X}$.

$\forall A \in \operatorname{ob}\mathfrak{X}$ DENOTE: $\mathrm{Res}(A) := $

$$\begin{tikzcd}
TA \arrow[r, shift left=2, "i_{TX}"] \arrow[r, shift right=2, "Ti_X"'] & T^2A \arrow[l, "m_X"'] \arrow[r, shift left=2, "i_{T^2X}"] \arrow[r, "Ti_{TX}" description] \arrow[r, shift right=2, "T^2i_X"'] & T^3A
\end{tikzcd}$$

IF $\mathfrak{X}$ HAS DESCENT OBJECTS:

$$\begin{tikzcd}
& TA \arrow[dr, "Ti_A"] \arrow[dd, phantom, "\Downarrow \varepsilon_A"] & \\
DA \arrow[ur, "e_A"] \arrow[dr, "e_A"'] & & T^2A \\
& TA \arrow[ur, "i_{TA}"'] &
\end{tikzcd}$$

CAN SHOW D IS A 2-MONAD

& $e : D \longrightarrow T$ 2-MONAD MORPHISM

„IDEMPOTENTIATION" OF A 1-MONAD:

- AFTER 0 STEPS: IF T ALREADY IDEMP. (SINCE $\eta_{TA} = T\eta_A$)
- AFTER 1 STEP: IF T PRESERVES COREFL. EQUALIZERS
- AFTER ∞ STEPS: IF $\mathcal{C}$ COMPL & WELL-POW YOU TAKE LIMIT OF: $T \hookleftarrow D_1 \hookleftarrow D_2 \hookleftarrow \ldots$

„LAX-IDEMPOTENTIATION" OF A 2-MONAD:

- ⇨ AFTER 0 STEPS: 💬
- AFTER 1 STEP: IF T PRESERVES COREFL. DESCENT OBJECTS
- AFTER ∞ STEPS: WIP

THM TFAE FOR (T, m, i) ON $\mathfrak{K}$:

- T IS LAX-IDEMPOTENT
- $\forall A \in \operatorname{ob} \mathfrak{K}$ THERE IS A 2-CELL

$$\begin{tikzcd}
 & TA \arrow[dr, "Ti_A"] \arrow[dd, phantom, "\Downarrow \lambda_A"] & \\
TA \arrow[ur, equal] \arrow[dr, equal] & & T^2A \\
 & TA \arrow[ur, "i_{TA}"'] &
\end{tikzcd}$$

THAT IS THE DESCENT OBJECT OF

$$\begin{tikzcd}
TA \arrow[r, shift left=3, "i_{TX}"] \arrow[r, shift right=3, "Ti_X"'] & T^2A \arrow[l, "m_X"'] \arrow[r, shift left=3, "i_{T^2X}"] \arrow[r, shift right=3, "T^2i_X"'] & T^3A \arrow[l, "Ti_{TX}"']
\end{tikzcd}$$

2.3 CHARACTERIZATION INVOLVING COLAX ADJUNCTIONS

$\boxed{2.3}$

DEF) A COLAX ADJUNCTION

$$(\Psi,\Phi):(\varepsilon,\eta): \begin{tikzcd} \mathcal{D} \arrow[r, bend right=40, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right=40, "F"'] \end{tikzcd}$$

CONSISTS OF
- F, U PSEUDOFUNCTORS
- $\varepsilon : FU \Rightarrow 1_{\mathcal{D}}$
  $\eta : 1_{\mathcal{C}} \Rightarrow UF$
  COLAX-NATURAL TRANSF.
- $$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FUF \arrow[d, "\varepsilon F"] \arrow[dl, phantom, "\overset{\Psi}{\Downarrow}"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\eta U"] \arrow[dr, equal] & UFU \arrow[d, "U\varepsilon"] \arrow[dl, phantom, "\overset{\Phi}{\Nearrow}"] \\ & U \end{tikzcd}$$
  MODIFS

SUBJECT TO AXIOMS

EXAMPLE AN OBJECT $I \in \mathrm{ob}\,\mathcal{D}$

GIVES
$$\begin{tikzcd} \mathcal{D} \arrow[r, bend right, "!"'] \arrow[r, phantom, "\perp"] & * \arrow[l, bend right, "I"'] \end{tikzcd}$$
IFF $\forall A \in \mathrm{ob}\,\mathcal{D} : \mathcal{D}(I, A)$

ADMITS INITIAL OBJECT

EXAMPLE $(\underline{\Psi}, \underline{\Phi}) : (\varepsilon, \eta) :$
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "1_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "T"] \end{tikzcd}$$

IMPLIES $\varepsilon_A \dashv \eta_A$ $\forall A$ W/ COUNIT
$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] \arrow[dr, phantom, shift left=4, "\underline{\Psi}_A \Swarrow"] & TA \arrow[d, "\varepsilon_A"] \\ & A \end{tikzcd}$$

DEF) GIVEN (T, m, i) ON $\mathcal{K}$,
DEFINE ITS KLEISLI 2-CATEGORY
AS FULL SUB-2-CAT'RY OF Ps-T-Alg
SPANNED BY FREE ALGEBRAS.

DENOTE $\mathcal{K}_T$

REMARK IS BIEQUIVALENT TO KLEISLI BICATEGORY
$\mathrm{Kl}(T)$ W/ OB: $\mathrm{ob}\,\mathcal{K}$
MOR: $f : A \rightsquigarrow B \equiv f : A \to TB \in \mathrm{mor}\,\mathcal{K}$

THM TFAE FOR (T, m, i) ON $\mathfrak{X}$:

- T IS LAX-IDEMPOTENT
- ANY BIADJUNCTION $$\begin{tikzcd} \mathfrak{X} \arrow[r, "F_T"'] & \mathfrak{X}_T \arrow[r, "G"'] & \mathcal{C} \arrow[ll, bend right, "H"', "\top"] \end{tikzcd}$$

INDUCES A COLAX ADJUNCTION

$$\begin{tikzcd} \mathfrak{X}_T \arrow[r, "G"'] & \mathcal{C} \arrow[l, bend right=60, "F_T H"', "\top"] \end{tikzcd}$$

WITH THE SAME COUNIT

PROOF: $\Leftarrow$

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "F_T"'] & \mathfrak{X}_T \arrow[r, "1_{\mathfrak{X}_T}"'] & \mathfrak{X}_T \arrow[ll, bend right, "U_T"', "\top"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathfrak{X}_T \arrow[r, "1_{\mathfrak{X}_T}"'] & \mathfrak{X}_T \arrow[l, bend right=60, "F_T U_T"', "\top"] \end{tikzcd}$$

COR TFAE FOR 1-MONAD (T, m, i) ON $\mathcal{C}$:

- T IS IDEMPOTENT
- ANY ADJUNCTION

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F_T"'] & \mathcal{C}_T \arrow[r, "G"'] & \mathcal{D} \arrow[ll, bend right=40, "H"'] \arrow[ll, bend right=20, phantom, "\top"]
\end{tikzcd}$$

INDUCES AN ADJUNCTION

$$\begin{tikzcd}
\mathcal{C}_T \arrow[r, "G"'] & \mathcal{D} \arrow[l, bend right=50, "F_T H"'] \arrow[l, bend right=25, phantom, "\top"]
\end{tikzcd}$$

WITH THE SAME COUNIT

2.4 AN ADDITIONAL IDEA

2.4

DEF) LET $U : \mathcal{D} \to \mathcal{C}$ A FUNCTOR,

$f : UA \to UB \in \operatorname{mor}\mathcal{C}$.

$$\begin{tikzcd}
UA \arrow[r, "f"] \arrow[dr, "U\sigma"'] & UB \arrow[d, "U\tau"] \\
 & UC
\end{tikzcd}$$

PAIR (σ, τ) IS U-COLIMIT OF f

IF $\forall (\mu, \nu)$ $\exists ! \theta$:

$$\begin{tikzcd}
UA \arrow[r, "f"] \arrow[dr, "U\sigma"'] \arrow[ddr, bend right, "\mu"'] & UB \arrow[dl, "U\tau"] \arrow[dd, bend left, "\nu"] \\
 & UC \arrow[d, dashed, "\theta"] \\
 & X
\end{tikzcd}$$

THM TFAE FOR (T, μ, η) ON $\mathcal{C}$:

- T IS IDEMPOTENT
- $\mathcal{C}_T$ ADMITS F_T-COLIMITS OF ARROWS

Thank you!