

# Lax adjunctions and lax-idempotent pseudomonads

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# Plan of the presentation

- ① Motivation,
- ② Colax adjunctions  $\longleftrightarrow$   $U$ -extensions,
- ③ Kleisli 2-category for a lax-idempotent pseudomonad,
- ④ Two-dimensional monad theory.

**Motto:** I make colax adjunctions.

# Motivation

# Motivation

Recall [BKP, 1989] – this paper concerns itself with a 2-monad  $T$  and its 2-categories  $T\text{-Alg}_s$  and  $T\text{-Alg}$ .

Examples of algebras and pseudo-morphisms include:

- monoidal categories and monoidal functors,
- small 2-categories and pseudofunctors,
- categories that admit  $\Phi$ -colimits and functors preserving them.

[BKP, 1989] proves their bicocompleteness, existence of various biadjunctions involving  $T\text{-Alg}$  ...

**Question:** What if we replace “pseudo” by “lax”?

Colax adjunctions  $\longleftrightarrow$   $U$ -extensions

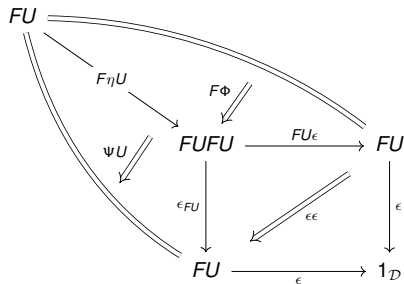
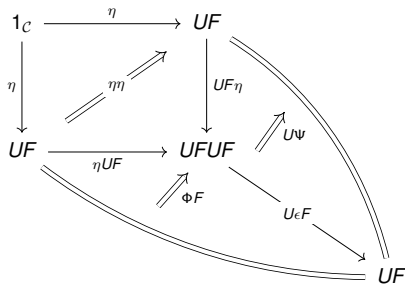
## Definition

Let  $\mathcal{C}, \mathcal{D}$  be 2-categories. A *colax adjunction* consists of two pseudofunctors  $U : \mathcal{D} \rightarrow \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$ , two colax natural transformations  $\eta : 1 \Rightarrow UF$  and  $\epsilon : FU \Rightarrow 1$  and two modifications:

$$\begin{array}{ccc}
 F & \xrightarrow{F\eta} & FUF \\
 & \searrow \psi & \downarrow \epsilon F \\
 & & F
 \end{array}
 \qquad
 \begin{array}{ccc}
 U & \xrightarrow{\eta U} & UFU \\
 & \searrow \phi & \downarrow U\epsilon \\
 & & U
 \end{array}$$

that are subject to the *swallowtail identities*:

the following have to be equal to the identity on  $\eta$  and  $\epsilon$  respectively:



We will denote the situation as follows:

$$(\Psi, \Phi) : (\epsilon, \eta) : F \dashv\vdash U : \mathcal{D} \rightarrow \mathcal{C}.$$

# Examples

## Example

A *biadjunction*: when  $\eta, \epsilon$  are pseudonatural and  $\Psi, \Phi$  are invertible. I will denote this by the usual symbol “ $\dashv$ ”.

## Example

Let  $\mathcal{K}$  be a 2-category. The 2-functor  $\mathcal{K} \rightarrow *$  admits a left colax adjoint if and only if there is an object  $L \in \mathcal{K}$  such that:

For every  $A \in \mathcal{K}$ ,  $\mathcal{K}(L, A)$  admits an initial object.



## Definition, [Bunge, 1974]

Let  $U : \mathcal{C} \rightarrow \mathcal{D}$  be a pseudofunctor,  $y_A : A \rightarrow UA'$ ,  $f : A \rightarrow UB$  1-cells of  $\mathcal{D}$ . The *left  $U$ -extension* of  $f$  along  $y_A$  is a pair  $(f', \psi)$  such that:

$$\begin{array}{c}
 A \xrightarrow{y_A} UA' \\
 \searrow f \quad \nearrow \psi \\
 \quad \quad \quad Uf' \downarrow \\
 \quad \quad \quad UB
 \end{array}
 \xRightarrow{U(\exists! \theta)}
 \begin{array}{c}
 A \xrightarrow{y_A} UA' \\
 \searrow f \quad \nearrow \forall \alpha \\
 \quad \quad \quad Uf' \downarrow \\
 \quad \quad \quad UB
 \end{array}
 =
 \begin{array}{c}
 A \xrightarrow{y_A} UA' \\
 \searrow f \quad \nearrow \forall \alpha \\
 \quad \quad \quad Uf' \downarrow \\
 \quad \quad \quad UB
 \end{array}$$

## Definition, [Bunge, 1974]

A collection of 1-cells  $\{y_A : A \rightarrow UA', A \in \mathcal{C}\}$  is *coherently closed for  $U$ -extensions* if:

- for every  $f : A \rightarrow UB$  we have a **choice** of a  $U$ -extension  $(f^{\mathbb{D}}, \mathbb{D}_f)$ ,
- the following is the left  $U$ -extension of  $f$  along  $y_X$ :

$$\begin{array}{ccccc}
 A & \xrightarrow{y_A} & UA' & & \\
 \downarrow f & & \downarrow U(y_{UX}f)^{\mathbb{D}} & & \\
 UX & \xrightarrow{y_{UX}} & U(UX)' \cong & \xrightarrow{U(1_{UX}^{\mathbb{D}} \circ (y_{UX}f)^{\mathbb{D}})} & \\
 & \uparrow \mathbb{D}_{1_{UX}} & \downarrow U1_{UX}^{\mathbb{D}} & & \\
 & & UX & & 
 \end{array}$$

The diagram illustrates the left  $U$ -extension of  $f$  along  $y_X$ . It shows a commutative square with  $A$  at the top-left,  $UA'$  at the top-right,  $UX$  at the bottom-left, and  $U(UX)'$  at the bottom-right. The horizontal arrows are  $y_A$  and  $y_{UX}$ . The vertical arrows are  $f$  and  $U(y_{UX}f)^{\mathbb{D}}$ . A curved arrow on the right indicates the composition  $U(1_{UX}^{\mathbb{D}} \circ (y_{UX}f)^{\mathbb{D}})$ . Below the square, there is a double arrow from  $UX$  to  $U(UX)'$  labeled  $\mathbb{D}_{1_{UX}}$ , and a vertical arrow from  $U(UX)'$  to  $UX$  labeled  $U1_{UX}^{\mathbb{D}}$ .

## Theorem

Let  $U : \mathcal{C} \rightarrow \mathcal{D}$  be a pseudofunctor,  $F : \text{ob } \mathcal{D} \rightarrow \text{ob } \mathcal{C}$  a function,  $\{y_A : A \rightarrow UFA, A \in \mathcal{C}\}$  a collection coherently closed for  $U$ -extensions. Assume that the following are  $U$ -extensions:

$$\begin{array}{ccc}
 A & \xrightarrow{y_A} & UFA \\
 & \searrow y_A & \downarrow \text{curly arrow} \\
 & & UFA
 \end{array}
 \quad
 \begin{array}{c}
 \text{curly arrow} \\
 \xRightarrow{\iota^{-1}} \\
 \text{curly arrow}
 \end{array}
 \quad
 U1_{FA}$$

$$\begin{array}{ccccc}
 A & \xrightarrow{y_A} & & UFA & \\
 f \downarrow & & \uparrow \mathbb{D} & & \downarrow U(y_B f)^{\mathbb{D}} \\
 B & \xrightarrow{y_B} & & UFB & \\
 g \downarrow & & \uparrow \mathbb{D} & & \downarrow U(y_C g)^{\mathbb{D}} \\
 C & \xrightarrow{y_C} & & UFC & 
 \end{array}$$

Then there is a pseudofunctor  $F : \mathcal{D} \rightarrow \mathcal{C}$ ,  $y : 1_{\mathcal{D}} \Rightarrow UF$  is colax natural and there is a colax adjunction:

$$(\Psi, \Phi) : (\epsilon, y) : F \dashv U : \mathcal{C} \rightarrow \mathcal{D}$$

## Theorem

Let  $(\Psi, \Phi) : (\epsilon, \gamma) : F \dashv\vdash U : \mathcal{C} \rightarrow \mathcal{D}$  be a colax adjunction between pseudofunctors in which  $\Psi$  is invertible.

Then the components of the unit  $\gamma_A : A \rightarrow UFA$  are coherently closed for  $U$ -extensions.

## Remark

The variation of these theorems where  $U$  is a 2-functor and  $F$  is a colax functor appeared in [Gray, 2006], [Bunge, 1974].

# Kleisli 2-category for a lax-idempotent pseudomonad

# Pseudomonads

## Definition

Let  $\mathcal{K}$  be a 2-category. A *pseudomonad* on  $\mathcal{K}$  consists of a pseudofunctor  $T : \mathcal{K} \rightarrow \mathcal{K}$ , pseudonatural transformations  $m : T^2 \Rightarrow T$ ,  $i : 1 \Rightarrow T$  and invertible modifications:

$$\begin{array}{ccc}
 T^3 & \xrightarrow{mT} & T^2 \\
 \downarrow Tm & \cong & \downarrow m \\
 T^2 & \xrightarrow{m} & T
 \end{array}$$

$$\begin{array}{ccccc}
 T & \xrightarrow{iT} & T^2 & \xleftarrow{Ti} & T \\
 & \searrow & \downarrow m & \swarrow & \\
 & & T & & 
 \end{array}
 \quad \cong$$

subject to axioms.

## Definition

Given a pseudomonad  $D$  on  $\mathcal{K}$ :

Denote by  $\text{Ps-}D\text{-Alg}$  the 2-category of pseudo- $D$ -algebras.

Denote by  $\mathcal{K}_D$  the *Kleisli 2-category*, defined as the full sub-2-category of  $\text{Ps-}D\text{-Alg}$  on free  $D$ -algebras.

## Definition

A pseudomonad  $(T, m, i)$  is said to be *lax-idempotent* if:

$$m_A \dashv i_{TA}, \text{ for all } A \in \mathcal{K},$$

with the counit of the adjunction  $m_A \circ i_{TA} \cong 1_{TA}$  being given by the invertible modification of the pseudomonad.

### Example - lax-idempotent pseudomonad

The *small presheaf pseudomonad*  $\mathcal{P} : \mathbf{CAT} \rightarrow \mathbf{CAT}$  sending a locally small category  $\mathcal{A}$  to the full subcategory of  $[\mathcal{A}^{op}, \mathbf{Set}]$  spanned by *small presheaves*.

The Kleisli 2-category  $\mathbf{CAT}_{\mathcal{P}} \simeq \mathbf{Prof}$ , the bicategory of locally small categories and *small profunctors*.

### Example - colax-idempotent 2-monad

Fix a 2-category  $\mathcal{K}$  with comma objects and  $C \in \mathcal{K}$ . There is a 2-monad  $T : \mathcal{K}/C \rightarrow \mathcal{K}/C$  whose pseudo- $T$ -algebras are fibrations in  $\mathcal{K}$ .

We have that:  $(\mathcal{K}/C)_T \cong \mathcal{K} // C$ , the *colax slice 2-category*.

### Example - lax-idempotent 2-comonad

Lax morphism classifier 2-comonad  $Q_l$  on the 2-category  $\mathbf{T-Alg}_s$ .



## The MAIN Colax Adjunction Theorem

Let  $D$  be a lax-idempotent pseudomonad on  $\mathcal{K}$ . Denote by  $J : \mathcal{K} \rightarrow \mathcal{K}_D$  the inclusion to the Kleisli 2-category. Any biadjunction as on the left induces a colax adjunction as on the right:

$$\begin{array}{ccc}
 \mathcal{K} & \xrightarrow{J} & \mathcal{K}_D \xrightarrow{G} \mathcal{L} \\
 & \nwarrow \scriptstyle H & \nearrow \scriptstyle \top \\
 & & \mathcal{L}
 \end{array}
 \rightsquigarrow
 \begin{array}{ccc}
 \mathcal{K}_D & \xleftarrow{JH} & \mathcal{L} \\
 & \nwarrow \scriptstyle \top & \nearrow \scriptstyle G \\
 & & \mathcal{L}
 \end{array}$$

### Proof

Consider the components of the counit  $s_L : GJ_DHL \rightarrow L$  of the biadjunction. They will also be the components of the counit of the colax adjunction. The proof consists of showing that they are coherently closed for  $G$ -lifts.

## Corollary 1

Given a lax-idempotent pseudomonad  $D$  on  $\mathcal{K}$ , the free-forgetful biadjunction induces a colax adjunction between the Kleisli 2-category and the 2-category of algebras:

$$\begin{array}{ccc}
 \mathcal{K} & \begin{array}{c} \xleftarrow{U^D} \\ \text{\tiny T} \\ \xrightarrow{F^D} \end{array} & \text{Ps-}D\text{-Alg} \\
 & \rightsquigarrow & \\
 \mathcal{K}_D & \begin{array}{c} \xleftarrow{J_D \circ U^D} \\ \text{\tiny T} \\ \xrightarrow{\quad} \end{array} & \text{Ps-}D\text{-Alg}
 \end{array}$$

## Remark

This is a categorification of the fact that for an idempotent 1-monad, the EM and Kleisli categories are equivalent.

## Corollary 2

Given a lax-idempotent pseudomonad  $D$  on a 2-category  $\mathcal{K}$ , the biadjunction between the base 2-category and the Kleisli 2-category induces a colax adjunction on the Kleisli 2-category:

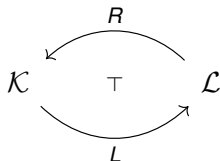
$$\begin{array}{ccc}
 \begin{array}{ccc}
 & F_D & \\
 \mathcal{K} & \xleftarrow{\quad} & \mathcal{K}_D \\
 & J_D & \\
 & \text{\tiny $\top$} & 
 \end{array}
 & \rightsquigarrow &
 \begin{array}{ccc}
 & J_D F_D & \\
 \mathcal{K}_D & \xleftarrow{\quad} & \mathcal{K}_D \\
 & \text{\tiny $\top$} & \\
 & \text{\tiny $\top$} & 
 \end{array}
 \end{array}$$

## Remark

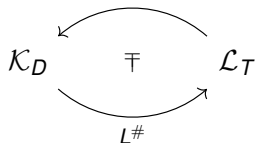
Colax adjoints are **not** unique up to an equivalence.

### Corollary 3 – change-of-base

Let  $D : \mathcal{K} \rightarrow \mathcal{K}$  be lax-idempotent pseudomonad and let  $T : \mathcal{L} \rightarrow \mathcal{L}$  be a pseudomonad. Assume that there is a biadjunction between the base 2-categories:



Assume that  $L$  extends to  $L^\# : \mathcal{K}_D \rightarrow \mathcal{L}_T$ . Then we have:



## Example

Given a 2-category  $\mathcal{K}$  with comma objects and pullbacks, recall that any 1-cell  $k : C \rightarrow D$  gives a 2-adjunction:

$$\begin{array}{ccc}
 & k^* & \\
 \mathcal{K}/C & \top & \mathcal{K}/D \\
 & k_* &
 \end{array}$$

By the previous theorem, this gives rise to a **lax** adjunction between the colax slice 2-categories:

$$\begin{array}{ccc}
 & & \\
 \mathcal{K} // C & \top & \mathcal{K} // D \\
 & k_* &
 \end{array}$$

# Weak limits

## Definition

Let  $\mathcal{K}$  be a 2-category and  $F : J \rightarrow \mathcal{K}$  a 2-functor. We say that a cone  $\lambda : \Delta L \rightarrow F$  exhibits  $L$  as a *coreflector-limit* of  $F$  if for every  $A \in \mathcal{K}$ :

$$\mathcal{K}(A, L) \xrightarrow{\kappa_A} \text{Cone}(A, F)$$

$$\theta \mapsto \theta \lambda$$

is a *coreflector* (admits a left adjoint with invertible unit).

## Remark

A variant of this called *quasi-colimits* were defined in [Gray, 2006]. Also this is a special case of an *enriched weak limit* of [LR, 2012].

# In elementary terms

A *coreflector-limit cone*  $\lambda : \Delta L \Rightarrow F$  satisfies that for every other cone  $\mu : \Delta A \Rightarrow F$ , there exists a map  $\theta_\mu : A \rightarrow L$  and an isomorphism  $\Phi_i$  as pictured below:

$$\begin{array}{ccc}
 A & \xrightarrow{\mu_i} & Fi \\
 \searrow \theta_\mu & \Downarrow \Phi_i & \nearrow \lambda_i \\
 & L &
 \end{array}$$

Moreover, given a 1-cell  $\theta : C \rightarrow A$  and a modification  $\sigma : \mu \rightarrow \theta\lambda$ , there exists a unique 2-cell  $\bar{\sigma} : \theta_\mu \Rightarrow \theta$  such that:

$$\begin{array}{ccc}
 A & \xrightarrow[\theta]{\mu_i} & Fi \\
 & \Downarrow \forall \sigma_i & \\
 A & \xrightarrow{\theta} C \xrightarrow{\lambda_i} & Fi
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow[\theta]{\mu_i} & Fi \\
 & \Downarrow \exists ! \bar{\sigma} & \\
 A & \xrightarrow{\theta_\mu} C \xrightarrow{\lambda_i} & Fi
 \end{array}$$

## Example

In case  $\kappa$  is an equivalence/isomorphism, we get *bilimits/2-limits*.

## Example

An object  $L$  in a 2-category  $\mathcal{K}$  is *coreflector-initial* if and only if:  
For every  $A \in \mathcal{K}$ ,  $\mathcal{K}(L, A)$  admits an initial object.

Consider the 2-category  $\text{MonCat}_l$  of monoidal categories and lax monoidal functors.  $*$  is coreflector-initial because for every monoidal category  $(\mathcal{A}, \otimes, I)$ :

$$\text{MonCat}_l(*, \mathcal{A}) \cong \text{Mon}(\mathcal{A}),$$

the category of monoids in  $\mathcal{A}$ .



## Corollary 4

Let  $D$  be a lax-idempotent pseudomonad on a 2-category  $\mathcal{K}$ .  
If  $\mathcal{K}$  admits  $J$ -indexed bilimits,  $\mathcal{K}_D$  admits them as coreflector-limits.

## Example

The bicategory  $\mathbf{Prof}$  of locally small categories and small profunctors is coreflector-complete.

# Two-dimensional monad theory

Let  $T$  be a 2-monad on a 2-category  $\mathcal{K}$ .

If  $T\text{-Alg}_s$  admits enough colimits, the inclusion  $T\text{-Alg}_s \rightarrow T\text{-Alg}_l$  admits a left 2-adjoint and in turn generates a 2-comonad  $Q_l : T\text{-Alg}_s \rightarrow T\text{-Alg}_s$  called the *lax morphism classifier 2-comonad*. Notice we have that  $(T\text{-Alg}_s)_{Q_l} \cong T\text{-Alg}_l$ .

If  $\mathcal{K}$  admits oplax limits of arrows,  $Q_l$  is lax-idempotent.

If these conditions are met, we will say that  $T$  satisfies **Property L**.

## Special Case of Main Colax Adjunction Theorem

Let  $T$  satisfy **Property L**. Any 2-adjunction below left induces a colax adjunction below right:

$$\begin{array}{ccc}
 & H & \\
 & \curvearrowright & \\
 T\text{-Alg}_S & \xrightarrow{J} T\text{-Alg}_I & \xrightarrow{G} \mathcal{L} \\
 & \perp & \\
 & \curvearrowleft &
 \end{array}
 \rightsquigarrow
 \begin{array}{ccc}
 & G & \\
 & \curvearrowright & \\
 T\text{-Alg}_I & \xrightarrow{JH} \mathcal{L} & \\
 & \perp & \\
 & \curvearrowleft &
 \end{array}$$

## Corollary 1

Let  $S, T$  be 2-monads satisfying **Property L** on a 2-category  $\mathcal{K}$ . Let  $\theta : S \rightarrow T$  be a 2-monad morphism. Assume that the induced 2-functor  $\theta^* : T\text{-Alg}_S \rightarrow S\text{-Alg}_S$  admits a left 2-adjoint. Then:

$$\begin{array}{ccc}
 & \curvearrowright & \\
 T\text{-Alg}_I & \overset{\overline{\tau}}{\curvearrowright} & S\text{-Alg}_I \\
 & \xrightarrow{\theta^*} &
 \end{array}$$

Recall the following:

### Theorem, [BKP, 1989]

Let  $T$  be a 2-monad on a 2-category satisfying **Property L**. If  $T\text{-Alg}_s$  is cocomplete,  $T\text{-Alg}$  is bicocomplete.

The 2-category of lax morphisms is seldom bicocomplete. But we have:

### Corollary 2

Let  $T$  be a 2-monad on a 2-category satisfying **Property L**. If  $T\text{-Alg}_s$  is cocomplete,  $T\text{-Alg}_l$  is coreflector-cocomplete.

## Example

The following 2-categories are coreflector-cocomplete:

- the 2-category of monoidal categories and lax-monoidal functors and its symmetric/braided variants,
- the 2-category of small 2-categories, lax functors, and icons,
- for a set  $\Phi$  of small categories, the 2-category  $\Phi\text{-Colim}_I$  of small categories that admit a choice of  $J$ -indexed colimits for  $J \in \Phi$  and **all** functors between them.

## Bonus: weak equalizers in Prof

What is the coreflector-equalizer of the following pair in Prof?

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad G \quad} & \mathcal{B} \\ & \xrightarrow{\quad H \quad} & \end{array}$$

Compute the equalizer in CAT of the following:

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{\quad E \quad} & \mathcal{P}\mathcal{A} & \begin{array}{c} \xrightarrow{\quad \mathcal{P}G \quad} \\ \xrightarrow{\quad \mathcal{P}H \quad} \end{array} & \mathcal{P}\mathcal{B} \end{array}$$

**Claim:** The weak limit in Prof given by transpose of  $E$ :

$$\mathcal{E} \xrightarrow{\quad \hat{E} \quad} \mathcal{A}$$

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Thank you.