Lax adjunctions and lax-idempotent pseudomonads

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Plan of the presentation

1. Motivation,
2. Colax adjunctions $\leftrightsquigarrow$ U-extensions,
3. Kleisli 2-category for a lax-idempotent pseudomonad,
4. Two-dimensional monad theory.

Motto: I make colax adjunctions.

Motivation

Motivation

Recall [BKP, 1989] – this paper concerns itself with a 2-monad T and its 2-categories T-Alg_s and T-Alg.

Examples of algebras and pseudo-morphisms include:

- monoidal categories and monoidal functors,
- small 2-categories and pseudofunctors,
- categories that admit Φ-colimits and functors preserving them.

[BKP, 1989] proves their bicocompleteness, existence of various biadjunctions involving T-Alg . . .

Question: What if we replace “pseudo” by “lax”?

Colax adjunctions $\leftrightsquigarrow$ U-extensions

Definition

Let $\mathcal{C}, \mathcal{D}$ be 2-categories. A *colax adjunction* consists of two pseudofunctors $U : \mathcal{D} \to \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$, two colax natural transformations $\eta : 1 \Rightarrow UF$ and $\epsilon : FU \Rightarrow 1$ and two modifications:

$$\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, equal, ""{name=A}] & FUF \arrow[d, "\epsilon F"] \arrow[to=A, Rightarrow, "\Psi"', shorten=2mm] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\eta U"] \arrow[dr, equal, ""{name=B}] & UFU \arrow[d, "U\epsilon"] \\
& U
\arrow[from=B, to=1-2, Rightarrow, "\Phi", shorten=2mm]
\end{tikzcd}$$

that are subject to the *swallowtail identities*:

the following have to be equal to the identity on η and ϵ respectively:

$$\begin{tikzcd}
1_{\mathcal{C}} \arrow[r, "\eta"] \arrow[d, "\eta"'] & UF \arrow[d, "UF\eta"] \arrow[ddr, bend left, equal] & \\
UF \arrow[r, "\eta UF"'] \arrow[ur, Rightarrow, "\eta\eta" description] \arrow[drr, bend right, equal] & UFUF \arrow[dr, "U\epsilon F"] \arrow[ur, phantom, "\Nearrow U\Psi"] \arrow[d, phantom, "\Nearrow \Phi F"] & \\
& & UF
\end{tikzcd}
\qquad
\begin{tikzcd}
FU \arrow[dr, "F\eta U"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\
\arrow[r, phantom, "\Swarrow \Psi U"] & FUFU \arrow[r, "FU\epsilon"] \arrow[d, "\epsilon FU"'] \arrow[ur, phantom, "F\Phi \Swarrow"] & FU \arrow[d, "\epsilon"] \arrow[dl, Rightarrow, "\epsilon\epsilon"] \\
& FU \arrow[r, "\epsilon"'] & 1_{\mathcal{D}}
\end{tikzcd}$$

We will denote the situation as follows:

$$(\Psi, \Phi) : (\epsilon, \eta) : F \dashv\!\!| \, U : \mathcal{D} \to \mathcal{C}.$$

Examples

Example

A *biadjunction*: when η, ϵ are pseudonatural and Ψ, Φ are invertible.

I will denote this by the usual symbol “$\dashv$”.

Example

Let $\mathcal{K}$ be a 2-category. The 2-functor $\mathcal{K} \to *$ admits a left colax adjoint if and only if there is an object $L \in \mathcal{K}$ such that:

For every $A \in \mathcal{K}$, $\mathcal{K}(L, A)$ admits an initial object.

Definition, [Bunge, 1974]

Let $U : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor, $y_A : A \to UA'$, $f : A \to UB$ 1-cells of $\mathcal{D}$. The *left U-extension* of f along y_A is a pair (f', ψ) such that:

$$\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[dr, "f"', ""{name=F}] & UA' \arrow[d, "Uf'"', ""{name=UF}] \arrow[d, bend left=70, "Ug", ""'{name=UG}] \\
 & UB
\arrow[from=F, to=1-2, Rightarrow, "\psi"']
\arrow[from=UF, to=UG, Rightarrow, "U(\exists!\theta)"]
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[dr, "f"', ""{name=F2}] & UA' \arrow[d, "Ug"] \\
 & UB
\arrow[from=F2, to=1-2, Rightarrow, "\forall\alpha"]
\end{tikzcd}$$

Definition, [Bunge, 1974]

A collection of 1-cells $\{y_A : A \to UA', A \in \mathcal{C}\}$ is *coherently closed for U-extensions* if:

- for every $f : A \to UB$ we have a **choice** of a U-extension $(f^{\mathbb{D}}, \mathbb{D}_f)$,
- the following is the left U-extension of f along y_X:

$$\begin{tikzcd}
A \arrow[rr, "y_A"] \arrow[d, "f"'] & & UA' \arrow[d, "U(y_{UX}f)^{\mathbb{D}}"] \arrow[dd, bend left=60, "U(1^{\mathbb{D}}_{UX} \circ (y_{UX}f)^{\mathbb{D}})"] \\
UX \arrow[rr, "y_{UX}"] \arrow[rrd, equal] & {} \arrow[u, Rightarrow, "\mathbb{D}_{y_{UX}f}"] & U(UX)' \cong \arrow[d, "U1^{\mathbb{D}}_{UX}"] \\
& {} \arrow[u, Rightarrow, "\mathbb{D}_{1_{UX}}"] & UX
\end{tikzcd}$$

Theorem

Let $U : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor, $F : \operatorname{ob} \mathcal{D} \to \operatorname{ob} \mathcal{C}$ a function, $\{y_A : A \to UFA, A \in \mathcal{C}\}$ a collection coherently closed for U-extensions. Assume that the following are U-extensions:

$$\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[dr, "y_A"'] & UFA \arrow[d, bend right=50, equal, ""{name=L}] \arrow[d, bend left=50, "U1_{FA}", ""'{name=R}] \\
 & UFA
\arrow[from=L, to=R, Rightarrow, "\iota^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathbb{D} \Uparrow"] & UFA \arrow[d, "U(y_B f)^{\mathbb{D}}"] \\
B \arrow[r, "y_B"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathbb{D} \Uparrow"] & UFB \arrow[d, "U(y_C g)^{\mathbb{D}}"] \\
C \arrow[r, "y_C"'] & UFC
\end{tikzcd}$$

Then there is a pseudofunctor $F : \mathcal{D} \to \mathcal{C}$, $y : 1_{\mathcal{D}} \Rightarrow UF$ is colax natural and there is a colax adjunction:

$$(\Psi, \Phi) : (\epsilon, y) : F \dashv U : \mathcal{C} \to \mathcal{D}$$

Theorem

Let $(\Psi, \Phi) : (\epsilon, y) : F \dashv U : \mathcal{C} \to \mathcal{D}$ be a colax adjunction between pseudofunctors in which Ψ is invertible.

Then the components of the unit $y_A : A \to UFA$ are coherently closed for U-extensions.

Remark

The variation of these theorems where U is a 2-functor and F is a colax functor appeared in [Gray, 2006], [Bunge, 1974].

Kleisli 2-category for a lax-idempotent pseudomonad

Pseudomonads

Definition

Let $\mathcal{K}$ be a 2-category. A *pseudomonad* on $\mathcal{K}$ consists of a pseudofunctor $T : \mathcal{K} \to \mathcal{K}$, pseudonatural transformations $m : T^2 \Rightarrow T$, $i : 1 \Rightarrow T$ and invertible modifications:

$$\begin{tikzcd}
T^3 \arrow[r, "mT"] \arrow[d, "Tm"'] \arrow[dr, phantom, "\cong"] & T^2 \arrow[d, "m"] \\
T^2 \arrow[r, "m"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "iT"] \arrow[dr, equal] \arrow[dr, phantom, "\cong", near start, shift left=6] & T^2 \arrow[d, "m" description] & T \arrow[l, "Ti"'] \arrow[dl, equal] \arrow[dl, phantom, "\cong", near start, shift right=6] \\
 & T &
\end{tikzcd}$$

subject to axioms.

Definition

Given a pseudomonad D on $\mathcal{K}$:

Denote by Ps-D-Alg the 2-category of pseudo-D-algebras.

Denote by $\mathcal{K}_D$ the *Kleisli 2-category*, defined as the full sub-2-category of Ps-D-Alg on free D-algebras.

Definition

A pseudomonad (T, m, i) is said to be *lax-idempotent* if:

$$m_A \dashv i_{TA}, \text{ for all } A \in \mathcal{K},$$

with the counit of the adjunction $m_A \circ i_{TA} \cong 1_{TA}$ being given by the invertible modification of the pseudomonad.

Example - lax-idempotent pseudomonad

The *small presheaf pseudomonad* $\mathcal{P} : \mathrm{CAT} \to \mathrm{CAT}$ sending a locally small category $\mathcal{A}$ to the full subcategory of $[\mathcal{A}^{op}, \mathrm{Set}]$ spanned by *small presheaves*.
The Kleisli 2-category $\mathrm{CAT}_{\mathcal{P}} \simeq \mathrm{Prof}$, the bicategory of locally small categories and *small profunctors*.

Example - colax-idempotent 2-monad

Fix a 2-category $\mathcal{K}$ with comma objects and $C \in \mathcal{K}$. There is a 2-monad $T : \mathcal{K}/C \to \mathcal{K}/C$ whose pseudo-T-algebras are fibrations in $\mathcal{K}$.
We have that: $(\mathcal{K}/C)_T \cong \mathcal{K}//C$, the *colax slice 2-category*.

Example - lax-idempotent 2-comonad

Lax morphism classifier 2-comonad Q_l on the 2-category T-Alg_s.

The MAIN Colax Adjunction Theorem

Let D be a lax-idempotent pseudomonad on $\mathcal{K}$. Denote by $J : \mathcal{K} \to \mathcal{K}_D$ the inclusion to the Kleisli 2-category. Any biadjunction as on the left induces a colax adjunction as on the right:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "J"'] & \mathcal{K}_D \arrow[r, "G"'] & \mathcal{L} \arrow[ll, bend right=40, "H"', "\top"]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathcal{K}_D \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\overline{\top}"] & \mathcal{L} \arrow[l, bend right=50, "JH"']
\end{tikzcd}$$

Proof

Consider the components of the counit $s_L : GJ_DHL \to L$ of the biadjunction. They will also be the components of the counit of the colax adjunction. The proof consists of showing that they are coherently closed for G-lifts.

Corollary 1

Given a lax-idempotent pseudomonad D on $\mathcal{K}$, the free-forgetful biadjunction induces a colax adjunction between the Kleisli 2-category and the 2-category of algebras:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend right, "F^D"'] \arrow[r, phantom, "\top"] & \text{Ps-}D\text{-Alg} \arrow[l, bend right, "U^D"'] \arrow[r, rightsquigarrow] & \mathcal{K}_D \arrow[r, hook, bend right] \arrow[r, phantom, "\overline{\top}"] & \text{Ps-}D\text{-Alg} \arrow[l, bend right, "J_D \circ U^D"']
\end{tikzcd}$$

Remark

This is a categorification of the fact that for an idempotent 1-monad, the EM and Kleisli categories are equivalent.

Corollary 2

Given a lax-idempotent pseudomonad D on a 2-category $\mathcal{K}$, the biadjunction between the base 2-category and the Kleisli 2-category induces a colax adjunction on the Kleisli 2-category:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend right, "J_D"'] \arrow[r, phantom, "\top"] & \mathcal{K}_D \arrow[l, bend right, "F_D"'] \arrow[r, phantom, "\rightsquigarrow"] & \mathcal{K}_D \arrow[r, bend right, equal] \arrow[r, phantom, "\mp"] & \mathcal{K}_D \arrow[l, bend right, "J_D F_D"']
\end{tikzcd}$$

Remark

Colax adjoints are **not** unique up to an equivalence.

Corollary 3 – change-of-base

Let $D : \mathcal{K} \to \mathcal{K}$ be lax-idempotent pseudomonad and let $T : \mathcal{L} \to \mathcal{L}$ be a pseudomonad. Assume that there is a biadjunction between the base 2-categories:

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, bend right, "L"'] \arrow[rr, phantom, "\top"] & & \mathcal{L} \arrow[ll, bend right, "R"']
\end{tikzcd}$$

Assume that L extends to $L^{\#} : \mathcal{K}_D \to \mathcal{L}_T$. Then we have:

$$\begin{tikzcd}
\mathcal{K}_D \arrow[rr, bend right, "L^{\#}"'] \arrow[rr, phantom, "\text{₸}"] & & \mathcal{L}_T \arrow[ll, bend right]
\end{tikzcd}$$

Example

Given a 2-category $\mathcal{K}$ with comma objects and pullbacks, recall that any 1-cell $k : C \to D$ gives a 2-adjunction:

$$\begin{tikzcd}
\mathcal{K}/C \arrow[r, bend right, "k_*"'] \arrow[r, phantom, "\top"] & \mathcal{K}/D \arrow[l, bend right, "k^*"']
\end{tikzcd}$$

By the previous theorem, this gives rise to a **lax** adjunction between the colax slice 2-categories:

$$\begin{tikzcd}
\mathcal{K}//C \arrow[r, bend right, "k_*"'] \arrow[r, phantom, "\mp"] & \mathcal{K}//D \arrow[l, bend right]
\end{tikzcd}$$

Weak limits

Definition

Let $\mathcal{K}$ be a 2-category and $F : J \to \mathcal{K}$ a 2-functor. We say that a cone $\lambda : \Delta L \to F$ exhibits L as a *coreflector-limit* of F if for every $A \in \mathcal{K}$:

$$\begin{tikzcd}
\mathcal{K}(A, L) \arrow[r, "\kappa_A"] & \mathrm{Cone}(A, F) \\
\theta \arrow[r, mapsto] & \theta\lambda
\end{tikzcd}$$

is a *coreflector* (admits a left adjoint with invertible unit).

Remark

A variant of this called *quasi-colimits* were defined in [Gray, 2006]. Also this is a special case of an *enriched weak limit* of [LR, 2012].

In elementary terms

A *coreflector-limit cone* $\lambda : \Delta L \Rightarrow F$ satisfies that for every other cone $\mu : \Delta A \Rightarrow F$, there exists a map $\theta_\mu : A \to L$ and an isomorphism Φ_i as pictured below:

$$\begin{tikzcd}
A \arrow[rr, "\mu_i", ""{name=T, below}] \arrow[dr, dashed, "\theta_\mu"'] & & Fi \\
& L \arrow[ur, "\lambda_i"'] \arrow[from=T, Rightarrow, dashed, "\Phi_i"'] &
\end{tikzcd}$$

Moreover, given a 1-cell $\theta : C \to A$ and a modification $\sigma : \mu \to \theta\lambda$, there exists a unique 2-cell $\overline{\sigma} : \theta_\mu \Rightarrow \theta$ such that:

$$\begin{tikzcd}
A \arrow[r, "\theta"'] \arrow[rr, bend left=50, "\mu_i", ""{name=U, below}] & C \arrow[r, "\lambda_i"'] \arrow[from=U, Rightarrow, "\forall\sigma_i"'] & Fi
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\theta_\mu", ""{name=M, below}] \arrow[r, bend right=50, "\theta"', ""{name=D, above}] \arrow[rr, bend left=50, "\mu_i", ""{name=V, below}] & C \arrow[r, "\lambda_i"'] \arrow[from=V, Rightarrow, "\Phi_i"'] & Fi \arrow[from=M, to=D, Rightarrow, "\exists!\overline{\sigma}"']
\end{tikzcd}$$

Example

In case κ is an equivalence/isomorphism, we get *bilimits/2-limits*.

Example

An object L in a 2-category $\mathcal{K}$ is *coreflector-initial* if and only if:
For every $A \in \mathcal{K}$, $\mathcal{K}(L, A)$ admits an initial object.

Consider the 2-category MonCat_l of monoidal categories and lax monoidal functors. $*$ is coreflector-initial because for every monoidal category $(\mathcal{A}, \otimes, I)$:

$$\mathrm{MonCat}_l(*, \mathcal{A}) \cong \mathrm{Mon}(\mathcal{A}),$$

the category of monoids in $\mathcal{A}$.

Corollary 4

Let D be a lax-idempotent pseudomonad on a 2-category $\mathcal{K}$.
If $\mathcal{K}$ admits J-indexed bilimits, $\mathcal{K}_D$ admits them as coreflector-limits.

Example

The bicategory Prof of locally small categories and small profunctors is coreflector-complete.

Two-dimensional monad theory

Let T be a 2-monad on a 2-category $\mathcal{K}$.

If T-Alg$_s$ admits enough colimits, the inclusion T-Alg$_s \to$ T-Alg$_l$ admits a left 2-adjoint and in turn generates a 2-comonad Q_l : T-Alg$_s \to$ T-Alg$_s$ called the *lax morphism classifier 2-comonad*. Notice we have that $(\text{T-Alg}_s)_{Q_l} \cong$ T-Alg$_l$.

If $\mathcal{K}$ admits oplax limits of arrows, Q_l is lax-idempotent.

If these conditions are met, we will say that T satisfies **Property L**.

Special Case of Main Colax Adjunction Theorem

Let T satisfy **Property L**. Any 2-adjunction below left induces a colax adjunction below right:

$$\begin{tikzcd}
\text{T-Alg}_s \arrow[r, "J"'] & \text{T-Alg}_l \arrow[r, "G"'] & \mathcal{L} \arrow[ll, bend right=25, "H"'] \arrow[ll, bend right=15, phantom, "\perp"] & \rightsquigarrow & \text{T-Alg}_l \arrow[r, "JH"'] & \mathcal{L} \arrow[l, bend right=50, "G"'] \arrow[l, bend right=25, phantom, "\underline{\perp}"]
\end{tikzcd}$$

Corollary 1

Let S, T be 2-monads satisfying **Property L** on a 2-category $\mathcal{K}$. Let $\theta : S \rightarrow T$ be a 2-monad morphism. Assume that the induced 2-functor $\theta^* : \text{T-Alg}_s \rightarrow \text{S-Alg}_s$ admits a left 2-adjoint. Then:

$$\begin{tikzcd}
\text{T-Alg}_l \arrow[rr, bend right=20, "\theta^*"'] & \overline{\top} & S\text{-Alg}_l \arrow[ll, bend right=30]
\end{tikzcd}$$

Recall the following:

Theorem, [BKP, 1989]

Let T be a 2-monad on a 2-category satisfying **Property L**. If T-Alg$_s$ is cocomplete, T-Alg is bicocomplete.

The 2-category of lax morphisms is seldom bicocomplete. But we have:

Corollary 2

Let T be a 2-monad on a 2-category satisfying **Property L**. If T-Alg$_s$ is cocomplete, T-Alg$_l$ is coreflector-cocomplete.

Example

The following 2-categories are coreflector-cocomplete:

- the 2-category of monoidal categories and lax-monoidal functors and its symmetric/braided variants,
- the 2-category of small 2-categories, lax functors, and icons,
- for a set Φ of small categories, the 2-category $\Phi\text{-Colim}_l$ of small categories that admit a choice of J-indexed colimits for $J \in \Phi$ and **all** functors between them.

Bonus: weak equalizers in Prof

What is the coreflector-equalizer of the following pair in Prof?

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left=2, "G", "|" marking] \arrow[r, shift right=2, "H"', "|" marking] & \mathcal{B}
\end{tikzcd}$$

Compute the equalizer in CAT of the following:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "E"] & \mathcal{P}\mathcal{A} \arrow[r, shift left=2, "\mathcal{P}G"] \arrow[r, shift right=2, "\mathcal{P}H"'] & \mathcal{P}\mathcal{B}
\end{tikzcd}$$

Claim: The weak limit in Prof given by transpose of E:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widehat{E}", "|" marking] & \mathcal{A}
\end{tikzcd}$$

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Thank you.