Quasi-limits and lax flexibility

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Plan of the presentation

1. Recall some definitions in 2-category theory,
2. Recall some results in two-dimensional monad theory,
3. Introduce lax versions of some concepts.

We will answer the following questions:

1. What kind of colimits do 2-categories of lax structures admit?
2. How to capture multicategories as coalgebras?

Some 2-category theory

Pseudomonads

Definition

A *pseudomonad* on a 2-category $\mathcal{K}$ consists of a pseudofunctor $T : \mathcal{K} \to \mathcal{K}$, pseudonatural transformations $m : T^2 \Rightarrow T$, $i : 1_{\mathcal{K}} \Rightarrow T$ and isomorphisms:

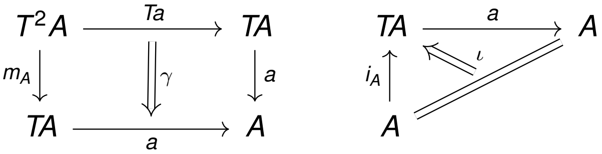
$$\begin{array}{ccc} T^3A & \xrightarrow{m_{TA}} & T^2A \\ {\scriptstyle Tm_A}\downarrow & \cong & \downarrow{\scriptstyle m_A} \\ T^2A & \xrightarrow[m_A]{} & TA \end{array} \qquad \begin{array}{ccccc} TA & \xrightarrow{i_{TA}} & T^2A & \xleftarrow{Ti_A} & TA \\ & \searrow\!\!= \;\; \cong & {\scriptstyle m_A}\downarrow & \cong \;\; =\!\!\swarrow & \\ & & TA & & \end{array}$$

Satisfying some higher associativity and unit laws.
If these isomorphism 2-cells are identities and m, i are 2-natural, we call it a *2-monad*.

Algebras

Definition

A *lax T-algebra* is a tuple (A, a, γ, ι) of an object A, a 1-cell $a : TA \to A$ and 2-cells γ, ι as pictured below:



These are subject to higher associativity and unit laws.

If γ, ι are invertible, it's a *pseudo-T-algebra*.
If γ, ι are the identities, it's a *strict T-algebra.*

Algebras - examples

Example

Free strict monoidal category 2-monad $M : \mathrm{Cat} \to \mathrm{Cat}$. It's given by:

$$M\mathcal{A} := \coprod_{n \geqslant 0} \mathcal{A}^n.$$

A strict T-algebra is a strict monoidal category. Pseudo-T-algebras are (equivalent to) ordinary monoidal categories.

Algebras - examples

Example

The *small presheaf pseudomonad* $\mathcal{P} : \mathrm{CAT} \to \mathrm{CAT}$.

$\mathcal{P}\mathcal{A}$ is defined as a full subcategory of $[\mathcal{A}^{op}, \mathrm{Set}]$ consisting of small presheaves. (A presheaf is said to be *small* if it's a small colimit of representables.)

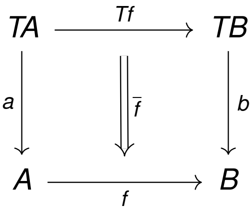
The unit $\mathcal{A} \to \mathcal{P}\mathcal{A}$ is given by the Yoneda embedding.

A pseudo-$\mathcal{P}$-algebra is a cocomplete category.

Algebra morphisms

Definition

A *lax morphism* of $(A, a) \rightsquigarrow (B, b)$ between T-algebras is a tuple $(f, \overline{f})$, where $f : A \to B$ is a morphism in $\mathcal{K}$ and $\overline{f}$ is a 2-cell:



Subject to higher associativity and unit laws.

If $\overline{f}$ is invertible, it's called a *pseudo* morphism.
If $\overline{f}$ is the identity, it's called a *strict* morphism.

Algebra morphisms

Example - free strict monoidal category 2-monad

Pseudo-M-morphism is a *monoidal functor*.

Example - free strict monoidal category 2-monad

Take two sup-semilattices A, B with a lowest element (that we denote by $\perp$).
Regard them as strict monoidal categories $(A, \vee, \perp)$, $(B, \vee, \perp)$.

Any order-preserving map $f : A \to B$ automatically lax monoidal because we have:

$$f(a) \vee f(b) \leqslant f(a \vee b)$$

Example - small presheaf pseudomonad

Pseudo-$\mathcal{P}$-morphism is a cocontinuous functor.

2-categories of algebras

We have the following 2-categories:

2-category	objects	morphisms
T-Alg_s	strict T-algebras	strict T-morphisms
T-Alg		pseudo T-morphisms
T-Alg_l		lax T-morphisms
Ps-T-Alg	pseudo T-algebras	pseudo T-morphisms

We also have inclusions:

$$\text{T-Alg}_s \to \text{T-Alg},$$
$$\text{T-Alg}_s \to \text{T-Alg}_l,$$
$$\text{T-Alg}_s \to \text{Ps-T-Alg},$$
$$\dots$$

Lax idempotent 2-monads

Definition

A 2-monad (T, m, i) is said to be *lax-idempotent* if there's an adjunction like this with the counit being the identity:

$$TA \mathrel{\substack{\xleftarrow{\;m_A\;} \\ \perp \\ \xrightarrow[\;i_{TA}\;]{}}} T^2A$$

In case this adjunction is an adjoint equivalence, we call T *pseudo-idempotent*.

Lax idempotent 2-monads

Proposition

Let (T, m, i) be a lax-idempotent 2-monad. The following are equivalent for an object $A \in \mathcal{K}$:

- A admits the structure of a pseudo-T-algebra,
- $i_A : A \to TA$ admits a left adjoint.

Example - lax-idempotent pseudomonad

The small presheaf pseudomonad $\mathcal{P}$.

Example - pseudo-idempotent 2-monad

The Cauchy completion 2-monad is pseudo-idempotent.

Two-dimensional monad theory

Two-dimensional monad theory I

Theorem, [Lack2002]

Let T be a 2-monad on $\mathcal{K}$. Assume T-Alg$_s$ admits reflexive iso-codescent objects. Then the inclusion has a left 2-adjoint:

$$\text{T-Alg}_s \underset{J}{\overset{(-)'}{\underset{\hookrightarrow}{\overset{\leftarrow}{\bot}}}} \text{T-Alg}$$

Denote the unit of this adjunction by $p_A : A \rightsquigarrow A'$ and the counit by $q_A : A' \to A$.

This adjunction generates a 2-comonad (Q_p, Qp, q) on T-Alg$_s$, call it the *pseudo-morphism classifier 2-comonad*.

Two-dimensional monad theory II

By the way, analogous story holds if we replace T-Alg by T-Alg_l - we obtain a *lax morphism classifier 2-comonad* Q_l:

Theorem, [Lack2002]

Let T be a 2-monad on $\mathcal{K}$. Assume T-Alg_s admits reflexive codescent objects. Then the inclusion has a left 2-adjoint:

$$\text{T-Alg}_s \underset{J}{\overset{(-)'}{\underset{\longleftarrow}{\overset{\perp}{\longleftarrow\!\!\!\!\longrightarrow}}}} \text{T-Alg}_l$$

Two-dimensional monad theory III

Theorem, [BKP1989]

Assume that $\mathcal{K}$ admits pseudo-limits of arrows. Then for every (A, a) we have:

$p_A \dashv q_A$ is an adjoint equivalence in T-Alg with $q_A p_A = 1$.

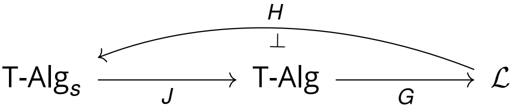
In particular, the pseudo-morphism classifier 2-comonad Q_p is pseudo-idempotent.

By the way, an analogous version holds for the lax morphism classifier 2-comonad Q_l - if $\mathcal{K}$ admits lax limits of arrows, it's colax-idempotent. See [LS2012].

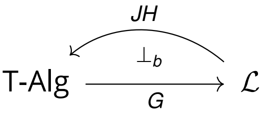
Two-dimensional monad theory IV

“A Big Biadjunction Theorem”, [BKP1989]

Any 2-adjunction as pictured below:



Induces a biadjunction:



Two-dimensional monad theory V

Corollary, [BKP1989]

Assume that T-Alg_s is 2-cocomplete. Then T-Alg is bicocomplete.

Question: What would be a lax version of this corollary? Is T-Alg_l bicocomplete if T-Alg_s is cocomplete?

Counterexample

The 2-category of strict monoidal categories and lax monoidal functors is **not** bicocomplete. It has no bi-initial object for instance.

Lax versions

How to properly generalize to the lax case?

Step 1: Notice that the theorem and its corollary holds in a little more generality:

Corollary

Let Q be a pseudo-idempotent 2-comonad on a 2-category $\mathcal{K}$ that is 2-cocomplete. Then the Kleisli 2-category $\mathcal{K}_Q$ is bicocomplete.

In case the 2-comonad Q is the pseudo-morphism classifier 2-comonad Q_p, we obtain the result about T-Alg being bicocomplete because T-Alg $= (\text{T-Alg}_s)_{Q_p}$.

Step 2: Instead of a pseudo-idempotent 2-comonad, let's take a colax-idempotent one.

Step 3: Define an appropriate lax version of bicolimits.

Quasi-colimits

Recall:

Definition

Let $\mathcal{K}$ be a 2-category and $F : J \to \mathcal{K}$ a 2-functor. We say that a cocone $\lambda : F \to \Delta C$ exhibits C as a *bicolimit* of F if the canonical 2-natural transformation (pictured below) is an equivalence for every $A \in \mathcal{K}$:

$$\begin{array}{ccc} \mathcal{K}(C, A) & \xrightarrow{\kappa_A} & \mathrm{Cocone}(F, A) \\ \theta & \mapsto & \theta\lambda \end{array}$$

Question: What if we only required that κ_A has a left or right adjoint? We get the notion of a *quasi-colimit*.

Quasi-colimits

Quasi-colimits were defined in the 70's in [Gray2006] but since then they have not really been studied (as far as I know).

Note that these are in particular *weak $\mathcal{V}$-colimits* (where $\mathcal{V} = \mathrm{Cat}$, $\mathcal{E} = \{\text{left adjoint functors}\}$) as studied in [SR2012].

I will focus on a certain special case of quasi-colimits.

Rali-colimits

Definition

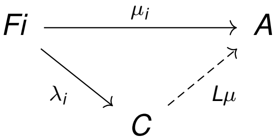
Let $\mathcal{K}$ be a 2-category and $F : J \to \mathcal{K}$ a 2-functor. We say that a cocone $\lambda : F \to \Delta C$ exhibits C as a *rali-colimit* of F if the canonical 2-natural transformation κ is a rali (right adjoint-left inverse) for each $A \in \mathcal{K}$:

$$\mathcal{K}(C, A) \underset{\kappa_A}{\overset{\exists L}{\rightleftarrows}} \mathrm{Cocone}(F, A) \qquad (L \perp \kappa_A)$$

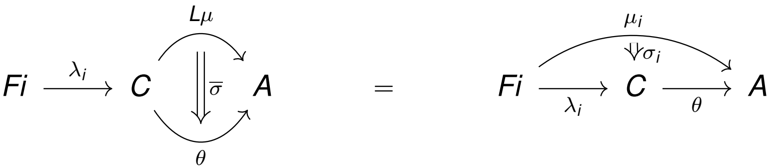
$$\theta \mapsto \theta\lambda$$

Rali-colimits

This in particular means that there is a *rali-colimit cocone* $\lambda : F \Rightarrow \Delta C$ such that for every other cocone $\mu : F \Rightarrow \Delta A$, there exists a map $L\mu$ pictured below:



Moreover, given a 1-cell $\theta : C \to A$ and a modification $\sigma : \mu \to \theta\lambda$, there exists a unique 2-cell $\overline{\sigma} : (L\mu) \Rightarrow \theta$ such that:



Examples I

Example

In a 2-category $\mathcal{K}$, I is a rali-initial object if for every $A \in \mathcal{K}$, the following map has a left adjoint:

$$\mathcal{K}(I, A) \xrightarrow{\quad ! \quad} *$$

This happens if and only if $\mathcal{K}(I, A)$ has an initial object for every I.

Examples II

Example

In the 2-category of strict monoidal categories and lax monoidal functors, the terminal monoidal category $*$ is rali-initial. This is because for every strict moncat $(\mathcal{A}, \otimes, I)$ we have:

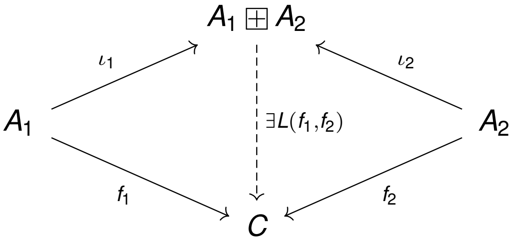
$$\mathsf{StrMonCat}_l(*, \mathcal{A}) \cong \mathsf{Mon}(\mathcal{A}).$$

For every monoidal category $\mathcal{A}$, the category of monoids in $\mathcal{A}$ has an initial object.

Examples III

Example

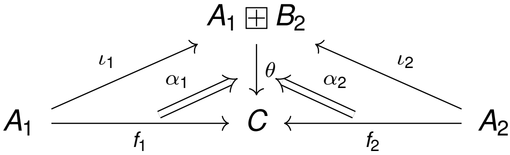
In a 2-category $\mathcal{K}$, given two objects A, B, their rali-coproduct is a triple $(\iota_1, \iota_2, A \boxplus B)$ such that for every (f_1, f_2) there exists a map as pictured below:



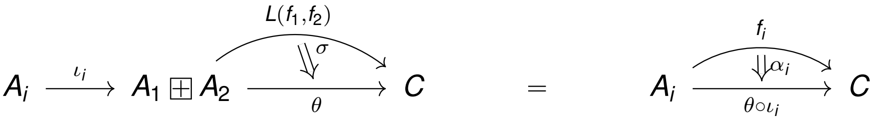
Examples IV

Example

Moreover, given a pair of 2-cells (α_1, α_2):



There exists a unique 2-cell $\sigma : L(f_1, f_2) \Rightarrow \theta$ satisfying ($i \in \{1, 2\}$):



Rali-cocompleteness

The Big Biadjunction Theorem has a lax analogue - "The Big Lax adjunction Theorem".

Corollary

Let Q be a colax-idempotent 2-comonad on a 2-category $\mathcal{K}$ that is 2-cocomplete. Then the Kleisli 2-category $\mathcal{K}_Q$ is rali-cocomplete.

The proof is very formal and admits various duals, for instance:

Corollary

Let T be a lax-idempotent 2-monad on a 2-category $\mathcal{K}$ that is 2-complete. Then the Kleisli 2-category $\mathcal{K}_T$ is rali-complete.

Rali-cocompleteness examples

Example: $Q = Q_l$ on T-Alg_s

Given a 2-monad T such that T-Alg_s is cocomplete, the result gives us that T-Alg_l is rali-cocomplete.

For instance, the 2-category of strict monoidal categories and **lax** monoidal functors has all rali-colimits.

Rali-cocompleteness examples

Example

Consider the small presheaf pseudomonad $\mathcal{P}$ on CAT.

A Kleisli morphism $\mathcal{A} \rightsquigarrow \mathcal{B}$ for this pseudomonad is a functor $F : \mathcal{A} \to \mathcal{P}\mathcal{B}$, It transposes to a *small profunctor*:

$$F : \mathcal{A} \times \mathcal{B}^{op} \to \mathrm{Set}.$$

$\mathrm{CAT}_{\mathcal{P}}$ is biequivalent to the bicategory Prof of locally small categories and small profunctors.

From the above result Prof admits all small rali-limits.

Flexibility

Flexibility

Recall again the adjunction below and the 2-comonad Q_p on T-Alg$_s$ it generates.

$$\text{T-Alg}_s \underset{J}{\overset{(-)'}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\perp}}}} \text{T-Alg}$$

Definition, [BKP1989]

A T-algebra (A, a) is *flexible* if the counit of the above adjunction $q_A : A' \to A$ is a retract-equivalence in T-Alg$_s$.

It is *semiflexible* if q_A is an equivalence in T-Alg$_s$.

Flexibility

Some properties:

- For a semiflexible (A, a), every pseudo-morphism $f : A \to B$ is isomorphic to a strict one,
- Flexible algebras are cofibrant objects for a certain model structure on T-Alg_s.

Proposition, [BG2013]

A T-algebra is semiflexible if and only if it has the structure of a pseudo-Q_p-coalgebra.

A T-algebra is flexible if and only if it has the structure of a **normal** pseudo-Q_p-coalgebra.

Flexibility

Definition, [BG2013]

Call a T-algebra *pie* if it has the structure of a **strict** Q_p-coalgebra.

Example

The following are equivalent for weights $W : J \to \mathrm{Cat}$:

- W is a pie algebra for the presheaf 2-monad on $[\mathrm{ob}\,\mathcal{J}, \mathrm{Cat}]$,
- W determines a PIE limit (limit built out of products, inserters, equifiers),
- the composite $\mathrm{ob} \circ W : \mathcal{J} \to \mathrm{Set}$ is a coproduct of representables (i.e. a *free presheaf*).

Flexibility

Example

A strict monoidal category $(\mathcal{A}, \otimes, I)$ is pie if and only if the underlying monoid on objects $(\mathrm{ob}\,\mathcal{A}, \otimes)$ is free.

Lax flexibility

Replacing Q_p by a general (colax-idempotent) 2-comonad Q, the study of semiflexible, flexible, pie algebras becomes the study of Q-coalgebras.

I will focus on what happens when $Q = Q_l$, the lax morphism classifier on T-Alg$_s$.

Definition

A T-algebra will be called *lax-pie* if it has the structure of a **strict** Q_l-coalgebra.

Intermezzo: Cat(T)

Let (T, μ, η) be a cartesian monad on $\mathcal{E}$.

It gives rise to a 2-monad Cat(T) : Cat$(\mathcal{E}) \to$ Cat$(\mathcal{E})$. Let's denote it by T again.

It sends a category $\mathcal{A}$ to a category $T\mathcal{A}$ such that:

$$\text{ob } T\mathcal{A} := T(\text{ob } \mathcal{A}),$$
$$\text{mor } T\mathcal{A} := T(\text{mor } \mathcal{A}).$$

Example

If T is a free monoid monad on Set, Cat(T) is the free strict monoidal category 2-monad on Cat.

Lax flexibility

Theorem

Let T be a cartesian monad on $\mathcal{E}$. There is an adjunction between T-Multicat and T-Alg_s:

$$T\text{-Multicat} \quad \underset{U}{\overset{F}{\leftrightarrows}} \;\perp\; \quad \text{T-Alg}_s$$

This adjunction generates the 2-comonad Q_l on T-Alg_s.

Lax flexibility

Theorem

The functor $U : \text{T-Multicat} \to \text{T-Alg}_s$ is comonadic.

Corollary

We have the following equivalence over T-Alg_s:

$$\text{T-Multicat} \simeq Q_l\text{-Coalg}_s.$$

Thus for a 2-monad T on Cat that comes from a cartesian monad T' on Set, lax-pie T-algebras are the same thing as T'-multicategories.

Examples

Example

A monoidal category $(\mathcal{A}, \otimes, I)$ is lax-pie if and only if there is a multicategory $\mathcal{M}$ such that $\mathcal{A}$ is a free monoidal category on $\mathcal{M}$.

Example, [DPP2006]

A double category X is lax-pie if and only if there is a *virtual double category* D such that X is a free double category associated to D.

Example

A 2-functor $W : \mathcal{J} \to \mathrm{Cat}$ is lax-pie if and only if there is a functor $\pi : \mathcal{E} \to \mathcal{J}$ such that: $W \cong (b \mapsto (\pi \downarrow b))$. These include weights for lax limits.

What more is there?

- Some characterizations of semiflexible algebras (pseudo-Q_p-coalgebras) studied in [BKP1989], [BG2013] generalize to characterizations of Q-coalgebras for a general colax-idempotent 2-comonad,
- Q_l-coalgebras for for $T = \mathrm{Cat}(T')$ are T'-multicategories. What happens for more general 2-monad T?

References I

- Bourke, John, and Richard Garner.
  On semiflexible, flexible and pie algebras
  Journal of Pure and Applied Algebra 217.2 (2013): 293-321.
- Blackwell, Robert, Gregory M. Kelly, and A. John Power.
  Two-dimensional monad theory.
  Journal of pure and applied algebra 59.1 (1989): 1-41.
- Dawson, RJ MacG, Robert Paré, and D. A. Pronk.
  Paths in double categories.
  Theory Appl. Categ. 16 (2006): 460-521.
- Gray, John Walker
  Formal category theory: adjointness for 2-categories.
  Vol. 391. Springer, 2006.
- Lack, Stephen, and Michael Shulman.
  Enhanced 2-categories and limits for lax morphisms.
  Advances in Mathematics 229.1 (2012): 294-356.

References II

Lack, Stephen.
Codescent objects and coherence.
Journal of Pure and Applied Algebra 175.1-3 (2002): 223-241.

Lack, Stephen, and Jiří Rosický.
Enriched weakness.
Journal of Pure and Applied Algebra 216.8-9 (2012): 1807-1822.

Weber, Mark
Familial 2-functors and parametric right adjoints.
Theory Appl. Categ 18.22 (2007): 665-732.

Thank you.